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Reserving

Unification of Stochastic Reserving Models Using Individual Claims Information

Eric Dal Moro[1a]

[1] Group Non-life Actuarial department, Baloise Insurance company

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In their daily reserving tasks, general insurance actuaries often use a mix of chain ladder, Bornhuetter-Ferguson, and Cape Cod methods to estimate the ultimate reserve amounts. These methods are usually applied on cumulative triangles and a payment or incurred pattern is derived from the application of the chain ladder method on these cumulative triangles. This pattern is then used in the Bornhuetter-Ferguson or Cape Cod method. Mack (2008) demonstrated that the stochastic model underlying this method should be based on incremental triangles. In addition, Saluz (2015) used a stochastic model for the Cape Cod method also based also on incremental triangles. Following on the works of Mack and Saluz, this paper will redevelop the chain ladder model on incremental triangles and unify the stochastic models of chain ladder, Bornhuetter-Ferguson, and Cape Cod into a single model. Based on this unified model, we will see that the first three moments (mean, variance and skewness) of the reserve risk distribution are defined by the relative position of the percentage of incremental claim to ultimate versus the claim pattern defining the best estimate.

Such study of the position of incremental claims versus best estimate pattern should be looked at on the individual claims level in order to have all the information of the claims development. The second part of this paper will therefore focus on the ways in which such pattern study on individual claims can be done. As a conclusion, the moments of the reserve risk distribution will be derived using individual claims information. A numerical example of such a study on individual claims will illustrate how the moments of the reserving risk distribution can be estimated.

Address for Correspondence: eric_dal_moro@yahoo.com

1. INTRODUCTION

When predicting the ultimate reserve amounts, general insurance actuaries use a mix of chain ladder, Bornhuetter-Ferguson (Bornhuetter and Ferguson 1972), and Cape Cod (Bühlmann and Straub 1983) methods. Most commercial reserving software systems propose these three methods. These systems will likely ask the application of chain ladder first to derive a payment or incurred pattern, which will then be used for the application of Bornhuetter-Ferguson or Cape Cod methods. In all the cases, the chain ladder method will be applied on a cumulative triangle. Below is a summary of the stochastic models underlying each of the three methods—chain ladder, Bornhuetter-Ferguson, and Cape Cod.

1.1. CHAIN LADDER

The chain ladder method is applied on **cumulative triangles**.

Let $C_{i,k}$ denote the cumulative claims amount (either paid or incurred) of accident year i after k years of development, $1 \leq i,\ k \leq n$ where n denotes the most recent

[a] Eric Dal Moro is Group P&C Chief Actuary at Baloise.
Eric has over 20 years of experience in reserving and risk management. Prior to Baloise, he worked in the consulting industry both in Paris and Zurich, and also for AXA in France, Japan and Italy and SCOR in different functions. He also gained some good experience in reinsurance when working at Swiss Re.
Eric has been serving as the Chairman of ASTIN Board of the International Actuarial Association since 2022 and between 2014 and 2017 and is an active researcher within the global actuarial community.

Table 1. Triangle per UWY and development year including ultimates

UWY	Dvpt					Ultimates
	1	2	3	4	5	
1	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	$C_{1,4}$	$C_{1,5}$	$\widehat{C}_{1,I}$
2	$C_{2,1}$	$C_{2,2}$	$C_{2,3}$	$C_{2,4}$		$\widehat{C}_{2,I}$
3	$C_{3,1}$	$C_{3,2}$	$C_{3,3}$			$\widehat{C}_{3,I}$
4	$C_{4,1}$	$C_{4,2}$				$\widehat{C}_{4,I}$
5	$C_{5,1}$					$\widehat{C}_{5,I}$
						$\sum_{j=1}^{I} \widehat{C}_{j,I}$

accident year. Then $C_{i,n+1-i}$ denotes the currently known claims amount of accident year i, shown in Table 1.

The basic chain ladder assumption is that there exist development factors $f_1, \ldots, f_{I-1}$ such that

$$E\left(C_{i,k+1} \mid C_{i,1}, \ldots, C_{i,k}\right) = f_k C_{i,k}, \quad 1 \leq i \leq I, \quad 1 \leq k \leq I-1 \tag{1}$$

where the link ratios (age-to-age factors) can be estimated as

$$\hat{f}_k = \frac{\sum_{j=1}^{I-k} C_{j,k+1}}{\sum_{j=1}^{I-k} C_{j,k}}, 1 \leq k \leq I-1, \tag{2}$$

under the assumption that $\{C_{i,1}, \ldots, C_{i,I}\}, \{C_{j,1}, \ldots, C_{j,I}\}, i \neq j$ are independent.

In this paper, $\hat{f}_k$ will denote the estimator of the random variable f_k. Mack (1993) shows that the link ratios $\hat{f}_k$ are unbiased and uncorrelated.

VARIANCE OF $C_{i,k}$

In the framework of the distribution-free calculation of the standard error of the reserve estimates, several variance models exist. For the purpose of this discussion, we will focus on the Mack standard error.

As for the variance of $C_{i,k+1}$, Mack (1993) induced that $\text{Var}\left(C_{i,k+1} \mid C_{i,1}, \ldots, C_{i,k}\right)$ (where $\text{Var}(A \mid B)$ denotes the conditional variance of A knowing B) should be proportional to $C_{i,k}$, i.e.:

$$\text{Var}\left(C_{i,k+1} \mid C_{i,1}, \ldots, C_{i,k}\right) = C_{i,k}\sigma_k^2, 1 \leq i \leq I, 1 \leq k \leq I-1 \tag{3}$$

where

$$\hat{\sigma}_k^2 = \frac{1}{I-k-1} \sum_{i=1}^{I-k} C_{i,k} \left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k\right)^2 \quad for \quad 1 \leq k \leq I-2 \tag{4}$$

It can be shown that the estimator $\hat{\sigma}_k^2$ is unbiased (Mack 1993).

1.2. BORNHUETTER-FERGUSON

As mentioned earlier, the Bornhuetter-Ferguson (hereinafter "BF") is usually applied on cumulative triangles using a pattern derived from the chain ladder method. In this section, we will review the stochastic model underlying the BF method introduced in Mack (2008). In this stochastic model, the BF method should be applied on **incremental triangles**.

As for the chain ladder method, let $C_{i,k}$ denote the cumulative claims amount (either paid or incurred) of accident year i after k years of development, $1 \leq i,\ k \leq n$ and ν_i be the premium volume of accident year i where n denotes the most recent accident year. Then $C_{i,n+1-i}$ denotes the currently known claims amount of accident year i. Let further $S_{i,k} = C_{i,k} - C_{i,k-1}$ denote the incremental claims amount (with $C_{i,0} = 0$) and U_i the (unknown) ultimate claims amount of accident year i. Then $R_i = U_i - C_{i,n+1-i}$ is the (unknown true) claims reserve for accident year i. Finally, let $S_{i,n+1} = U_i - C_{i,n}$ be the incremental claims amount after development year n (tail development).

Bornhuetter and Ferguson (1972) introduced their method to estimate R_i as follows:

$$\widehat{R}_i^{BF} = \widehat{U}_i\left(1 - \hat{z}_{n+1-i}\right)$$

where $\widehat{U}_i = \nu_i \hat{q}_i$ with a prior estimate $\hat{q}_i$ for the ultimate claims ratio $q_i = \frac{U_i}{\nu_i}$ of accident year i, $\hat{z}_k \in [0;1]$ is the estimated percentage of the ultimate claims amount which is expected to be known after development year k.

The BF stochastic model developed in Mack (2008) relies on the following assumptions related to the increments $S_{i,k}$:

- BF1: All increments $S_{i,k}$ are independent
- BF2: There are unknown parameters x_i, y_k such that:
  - $E\left(S_{i,k}\right) = x_i y_k$
  - $y_1 + \ldots + y_{n+1} = 1$
- BF3: There are unknown proportionality constants s_k^2 with $Var\left(S_{i,k}\right) = x_i s_k^2$

On the basis of these three assumptions, the prediction error of Bornhuetter-Ferguson can be estimated (Mack 2008). The prediction error, usually denoted as MSEP (mean squared error of prediction) consists of two components, the process error and the estimation error. Whereas the estimation error basically always can be calculated via the laws of error propagation, for the process error a stochastic model of the claims process was developed by Mack (2008).

Following Mack (2008), we have the following with x_1,… x_n known:

$$\hat{y}_k = \frac{\sum_{i=1}^{n+1-k} S_{i,k}}{\sum_{i=1}^{n+1-k} x_i}$$

is a best linear unbiased estimate of y_k and

$$\hat{s}_k^2 = \frac{1}{n-k} \sum_{i=1}^{n+1-k} \frac{\left(S_{i,k} - x_i \hat{y}_k\right)^2}{x_i}$$

is an unbiased estimate of $s^2{}_k$.

1.3. CAPE COD

As for the chain ladder and BF methods, we denote the cumulative claims (cumulative payments or incurred losses) in accident year $i \in \{0, \ \ldots, I\}$ at the end of development year $j \in \{0, \ \ldots, J\}$ by $C_{i,j} > 0$ and we assume J ⩽ I. Let $S_{i,j} = C_{i,j} - C_{i,j-1}$ denote the incremental claims, where we set $C_{i,-1} = 0$. The summation over an index starting from 0 is denoted with a square bracket, for example:

$$C_{\lfloor k \rfloor, j} = \sum_{i=0}^{k} C_{i,j}, \ 0 \le k \le I, \ 0 \le j \le J.$$

We assume that all claims are settled after development year J and therefore the total ultimate claim of accident year i is given by $C_{i,J}$. At time I, we have information in the upper left trapezoid/triangle:

$$D_I = \{C_{i,j} : i + j \le I, \ j \le J\}$$

and our goal is to predict the lower right triangle:

$$D_I^c = \{C_{i,j} : i + j > I, \ i \le I, j \le J\}.$$

The chain ladder prediction of the ultimate claim $C_{i,J}$ of accident year i > I − J is given by

$$\widehat{C}_{i,J}^{CL} = C_{i,\iota(i)} \prod_{j=\iota(i)}^{J-1} \widehat{f}_j$$

where

$$\hat{f}_j = \frac{C_{[I-j-1],j+1}}{C_{[I-j-1],j}}$$

and $\iota(i) = \min(J, I-i)$.

The chain ladder development pattern is defined as

$$\widehat{\beta}_j^{CL} = \prod_{k=j}^{J-1} \widehat{f}_k^{-1}, 0 \le j \le J-1, \ \widehat{\beta}_J^{CL} = 1 \tag{7}$$

The Cape Cod predictor (Bühlmann and Straub 1983) for the ultimate claim is given by

$$\widehat{C}_{i,J}^{CC} = C_{i,\iota(i)} + \upsilon_i \hat{q} \left(1 - \widehat{\beta}_{\iota(i)}\right)$$

where

The earned premium for accident year i is denoted by υ_i; and

$$\hat{q} = \frac{\sum_{i=0}^{I} C_{i,\iota(i)}}{\sum_{i=0}^{I} \upsilon_i \widehat{\beta}_{\iota(i)}}.$$

$\widehat{\beta}_{\iota(i)}$ is an estimate of $\beta_{\iota(i)}$ and describes the percentage of claims emerging up to development year $\iota(i)$. The incremental development pattern $\gamma_j = \beta_j - \beta_{j-1}$ is estimated by

$$\widehat{\gamma}_0 = \widehat{\beta}_0$$

$$\widehat{\gamma}_{j+1} = \widehat{\beta}_{j+1} - \widehat{\beta}_j, \quad 0 \le j \le J-1.$$

In the original article of Bühlmann and Straub (1983), it is mentioned that the estimation of the development pattern $\widehat{\beta}_j$ is an unsolved problem. In practice, the development pattern is often estimated by the chain ladder (CL) development pattern given in (7).

Finally, we define the outstanding loss liabilities for accident year i at time I as:

$$R_i^{CC} = C_{i,J} - C_{i,I-i}$$

MODEL ASSUMPTIONS

Incremental claims $S_{i,j}$ are independent and there exist positive parameters q, t_k^3, $0 \le j \le J$ and a development pattern $\gamma_0, \ \ldots, \ \gamma_J$ with $\sum_{j=0}^{J} \gamma_j = 1$ such that

$$E\left[S_{i,j}\right] = \upsilon_i q \gamma_j$$

$$Var\left[S_{i,j}\right] = (\upsilon_i q) \ \sigma_j^2$$

where $Var\left[S_{i,j}\right]$ denotes the variance of the random variable $S_{i,j}$.

For the estimation of the variance, we need estimates for $q\,\sigma_j^2$. Note that

$$\widehat{q\,\sigma_j^2} = \frac{1}{I-j} \sum_{i=0}^{I-j} \frac{1}{\nu_i} \left(S_{i,j} - \upsilon_i \widehat{\gamma}_j\right)^2, \tag{8}$$

$$0 \le j \le J, \ j \ne I$$

is an unbiased estimator for $q\,\sigma_j^2$.

Note also that the above model assumptions assume that the expected loss ratio q is the same for all accident years.

2. A REVIEW OF THE CHAIN LADDER METHOD

In this section, we are going to review the equation (4) of the chain ladder method:

$$\hat{\sigma}_k^2 = \frac{1}{I-k-1} \sum_{i=1}^{I-k} C_{i,k} \left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k\right)^2 \text{ for}$$

$$1 \le k \le I-2.$$

Using the definition $S_{i,k+1} = C_{i,k+1} - C_{i,k}$ and the chain ladder incremental pattern $\widehat{w}_k = \widehat{\beta}_{k-1}^{CL} - \widehat{\beta}_k^{CL}$, we can reformulate equation (4) as

$$\widehat{\sigma}_k^2 = \frac{1}{I-k-1} \sum_{i=1}^{I-k} C_{i,k} \left(\frac{S_{i,k+1}}{C_{i,k}} - \left(\widehat{f}_k - 1\right)\right)^2.$$

We can see that

$$\widehat{w}_k = \widehat{\beta}_{k-1}^{CL} - \widehat{\beta}_k^{CL}$$

$$= \frac{1}{\prod_{l=k-1}^{J-1} \widehat{f}_l} - \frac{1}{\prod_{l=k}^{J-1} \widehat{f}_l}$$

$$= \frac{1}{\prod_{l=k}^{J-1} \widehat{f}_l} \left(\widehat{f}_k - 1\right).$$

Hence,

$$\widehat{\sigma}_k^2 = \frac{1}{I-k-1} \sum_{i=1}^{I-k} C_{i,k} \left(\frac{S_{i,k+1}}{C_{i,k}} - \widehat{w}_k \prod_{l=k}^{J-1} \widehat{f}_l\right)^2.$$

As a result,

$$\widehat{\sigma}_k^2 = \frac{1}{I-k-1} \sum_{i=1}^{I-k} \widehat{C}_{i,J}^{CL} \prod_{l=k}^{J-1} \widehat{f}_l \left(\frac{S_{i,k+1}}{\widehat{C}_{i,J}^{CL}} - \widehat{w}_k\right)^2.$$

The shape of this equation is very similar to the volatility factor of the Bornhuetter-Ferguson and Cape Cod methods. Despite its unusual shape, it is the same as the usual known equation (4) used to estimate the $\widehat{\sigma}_k^2$ but based on incremental triangle.

As a conclusion of this section, we can mention that, in practice, actuaries usually estimate their loss ultimates on the basis of cumulative triangles. However, as we have seen, the stochastic underlying models are based on incremental claims. Incremental claims have the advantage of having independence between each triangle cell, which is not the case for cumulative claims. This is why the stochastic underlying models are based on increments. It would therefore be advisable to change the commercial reserving software and provide actuaries with analysis based on incremental claims: this would better reflect the dynamics of the claims movements.

3. STOCHASTIC RESERVING METHODS—A UNIFICATION

Equations (6), (8) and (9) are rewritten below.

BORNHUETTER-FERGUSON

$$\hat{s}_k^2 = \frac{1}{n-k}\sum_{i=1}^{n+1-k} x_i\left(\frac{S_{i,k}}{x_i} - \hat{y}_k\right)^2$$

where x_i represents the a-priori ultimate of the BF method.

The stochastic model defines: $Var\left(S_{i,k}\right) = x_i s_k^2$.

CAPE COD

$$\widehat{q\,\sigma_j^2} = \frac{1}{I-j}\sum_{i=0}^{I-j} \upsilon_i\left(\frac{S_{i,j}}{\nu_i} - \widehat{\gamma}_j\right)^2$$

where υ_i represents the ultimate premium of the Cape-Cod method.

The stochastic model defines: $Var\left[S_{i,j}\right] = (\upsilon_i q)\ \sigma_j^2$.

CHAIN LADDER

$$\widehat{\sigma}_k^2 = \frac{1}{I-k-1}\sum_{i=1}^{I-k} \widehat{C}_{i,J}^{CL}\prod_{l=k}^{J-1}\widehat{f}_l\left(\frac{S_{i,k+1}}{\widehat{C}_{i,J}^{CL}} - \widehat{w}_k\right)^2$$

where $\widehat{C}_{i,J}^{CL}$ represents the prediction of the ultimate claim $C_{i,J}$.

The stochastic model defines

$$Var\left(S_{i,k+1}|C_{i,1},\ldots,C_{i,k}\right) = C_{i,k}\sigma_k^2$$

The variance of the increment for each model is defined according to these volatility factors. These three equations have the same shape:

- The volatility factor always depends on the difference between the increments divided by the ultimate and the estimated pattern.
- The volatility factor is always a weighted average of these differences where the weights are either the ultimates (BF, Cape Cod) or derived from the ultimates (chain ladder).

In relation to the first point, in practice, the feeling for the volatility of a line of business always depends on the possibility for the actuary to have confidence in the incurred/payment pattern. When an actuary feels unsure about the incurred/payment pattern, he will say that the line of business is very volatile. On the opposite when the incurred/payment pattern is stable across the accident or underwriting years, he will say that the line of business is not volatile. The three equations reflect therefore the practice. In addition, their similarity shows that the applied method is not a determinant of the volatility of the resulting ultimates.

As for the skewness factors, the definitions are provided below.

BORNHUETTER-FERGUSON (DAL MORO 2021)

On the basis of the three assumptions described in 1.b (All increments $S_{i,k}$ are independent, there are unknown parameters x_i, y_k, there are unknown proportionality constants s_k^2 with $Var\left(S_{i,k}\right) = x_i s_k^2$), the prediction error of Bornhuetter-Ferguson can be estimated (Mack 2008). In order to estimate the skewness of the BF method, we need a fourth assumption:

BF4: There are unknown proportionality constants t_k^3 with SK$(S_{i,k}) = x_i^{\frac{3}{2}} t_k^3$

and

$$\hat{t}_k^3 = \frac{1}{n-k}\sum_{i=1}^{n+1-k}\frac{\left(S_{i,k}\ -\ x_i\hat{y}_k\right)^3}{x_i^{\frac{3}{2}}}$$

It has to be noted that the skewness of Bornhuetter-Ferguson in the proposed model comes in a distribution-free environment. Once the best estimate, the standard deviation and the skewness of the reserves are estimated, the actuary can fit a distribution of his choice to these first three moments.

CAPE COD (DAL MORO 2022)

As for the skewness of Cape Cod in a distribution free environment, following on the work of Saluz (2015), we assume that there exist positive parameters q, t_j^3, $0 \le j \le J$ and a development pattern $\gamma_0,\ \ldots,\ \gamma_J$ with $\sum_{j=0}^{J}\gamma_j = 1$ such that

$$E\left[X_{i,j}\right] = \upsilon_i q \gamma_j$$
$$SK\left[X_{i,j}\right] = (\upsilon_i q)^{\frac{3}{2}}\ t_j^3$$

where $SK\left[X_{i,j}\right]$ denotes the third moment of the random variable $X_{i,j}$. And υ_i corresponds to the UWY premium and q to the Cape Cod loss ratio.

For the estimation of the skewness, we need estimates for $q^{\frac{3}{2}}\ t_j^3$. Note that

$$\widehat{q^{\frac{3}{2}}\ t_j^3} = \frac{1}{I-j}\sum_{i=0}^{I-j}\frac{1}{\upsilon_i^{\frac{3}{2}}}\left(X_{i,j} - \upsilon_i\widehat{\gamma}_j\right)^3,$$
$$0 \le j \le J,\ j \ne I$$

is an unbiased estimator for $q^{\frac{3}{2}}\ t_j^3$.

As for Bornhuetter-Ferguson, it has to be noted that the skewness of Cape Cod in the proposed model comes in a distribution-free environment. Once the best estimate, the standard deviation and the skewness of the reserves are estimated, the actuary can fit a distribution of his choice to these first three moments.

CHAIN LADDER (DAL MORO 2013)

$$SK\left(C_{i,k+1} \mid C_{i,1}, \ldots, C_{i,k}\right) = C_{i,k}^{3/2} Sk_k^3, 1 \leq i \leq I, 1 \leq k \leq I-2$$

where:

$$\hat{S}k_k^3 = \frac{1}{\left(I-k-\frac{\left(\sum_{i=1}^{I-k} C_{i,k}^{3/2}\right)^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3}\right)} \sum_{i=1}^{I-k} C_{i,k}^{3/2}\left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k\right)^3 \quad (10)$$

for $1 \leq k \leq I-3$

It has to be noted that, like for Bornhuetter-Ferguson and Cape Cod, the skewness of chain ladder in the proposed model comes in a distribution-free environment. Once the best estimate, the standard deviation, and the skewness of the reserves are estimated, the actuary can fit a distribution of his choice to these first three moments.

Like for the volatility factor, the chain ladder skewness factor can be changed to reflect the difference between the increments divided by the ultimate and the estimated pattern.

The skewness factors show, as they do for the volatility factors, a unity in terms of the general shape of the formulae.

As a **unification of these three methods**, for the volatility and skewness estimators denoted respectively v_k^2 and sk_k^3, we could therefore consider the overall equations:

$$v_k^2 = \frac{1}{m(k)} \sum_i g\left(\widehat{U}_l\right)\left(\frac{S_{i,k}}{\widehat{U}_i} - \hat{y}_k\right)^2 \quad (11)$$

where $m(k)$ represents either I-k or I-k-1, $\widehat{U_i}$ represents the ultimate, and $g\left(\widehat{U_i}\right)$ is a function of the ultimate.

$$sk_k^3 = \frac{1}{o(k)} \sum_i h\left(\widehat{U}_l\right)\left(\frac{S_{i,k}}{\widehat{U}_i} - \hat{y}_k\right)^3 \quad (12)$$

with the same variables as for the volatility and o(k) is I-k or the factor before the sum in equation (10).

4. INDIVIDUAL CLAIMS ANALYSIS FOR A UNIFIED MODEL

Based on the unified notation described above (equations 11 and 12), we see that the reserve risk distribution is defined by the relative position of the percentage of incremental claim to ultimate vs. the pattern that is defining the best estimate. As mentioned earlier, in practice, this is the work of the actuary to try and defend his choice of best estimate patterns. Nowadays, such study of the position of incremental claims vs. best estimate pattern should be looked at individual claims level.

For this purpose, let's denote $SIC_{p,i,k}$ the individual incremental claim amount for claim p for accident year i at the end of development year k and $\widehat{UIC}_{p,i,k}$ the ultimate claim for claim p for the same accident year and development year. We have then:

$$\frac{S_{i,k}}{\widehat{U}_i} = \frac{\sum_p SIC_{p,i,k}}{\sum_p \widehat{UICC}_{p,i,k}} = \sum_p \frac{SIC_{p,i,k}}{\widehat{UICC}_{p,i,k}} \frac{\widehat{UIC}_{p,i,k}}{\sum_p \widehat{UIC}_{p,i,k}}$$

Let's denote

$$a_{p,i,k} = \frac{\widehat{UICC}_{p,i,k}}{\sum_p \widehat{UIC}_{p,i,k}}$$

Then we have the following (as $\sum_p a_{p,i,k} = 1$):

$$\frac{s_{i,k}}{\widehat{U}_i} - \hat{y}_k = \sum_p a_{p,i,k}\left(\frac{SIC_{p,i,k}}{\widehat{UTCC}_{p,i,k}} - \hat{y}_k\right). \quad (13)$$

With this last equation, we can see that some information related to the unified volatility and skewness estimated in equations (11) and (12) can be derived from individual claims. In equation (13), for i and k given, the following elements are not random: $a_{p,i,k}$, $\widehat{UIC}_{p,i,k}$ *and* $\hat{y}_k$. Therefore, the risk distribution depends on the shape (volatility and skewness) of the individual claims in year i and development year k.

In this context, let's look at an individual claims triangle and see the consequences of looking at the reserving distribution on such a triangle, shown in Table 2.

Any projection method applied to this triangle (e.g., neural networks, chain ladder ...) will just estimate the IBNER (incurred but not enough reported) and will not take into account the IBNYR (incurred but not yet reported). Therefore, in the context of a unified reserving model based on an individual claims triangle, the following procedure will have to be followed:

1. Estimate the future number of claims and the volatility and skewness of the distribution of the future number of claims. This can be done on using a chain ladder method (Mack 1993; Dal Moro 2013), shown in Table 3.
2. Estimate for each development year, the mean, volatility and skewness of individual claim incurred or payment. In this regard, we assume that the exposures remain stable across the UWYs allowing the calculation at development year level and not at development and UWY level.
3. Having the future number of claims and the individual incurred or payment per development year, it is easy to build the IBNYR risk distribution (see following sections for details).
4. As for the IBNER, we will rely on Schnieper (1991), which gives a complete description of the way in which IBNER and their volatilities can be estimated based on individual claim information. In a following section, we will extend the calculations to the skewness case.

In the next two sections, we are going to detail the four steps just described. It has to be noted that the most significant amount of reserves will come from the IBNYR as, for each new claim, the full mean ultimate payment has to be reserved.

5. IBNYR BASED ON INDIVIDUAL CLAIMS ANALYSIS

Based on the individual claims' triangle of Table 2, we can estimate the following parameters:

Table 2. Individual claim triangle (example)

Claim amount		Development year k			
UWY i	Claim ref	1	2	3	4
1	1	10	12	13	13
	2		8	8	
	3	20	23	24	25
	4		33	34	34
	5	15	17	18	19
2	6		5	6	
	7	40	42	50	
	8	7	9		
3	9	50	52		
	10	5	9		
	11	4			
4	12	8			
	13	12			

Table 3. Number of claims (from Table 2)

Nb claims	Development year k			
UWY i	1	2	3	4
1	3	5	5	4
2	2	3	2	
3	3	2		
4	2			

- The mean incurred/payment per development year;
- The variance of the incurred/payment per development year;
- The skewness of the incurred/payment per development year.

Having estimated the above characteristics, we can use the law of total variance to estimate the overall variance for accident year i:

$$Var\left(\sum_{p=1}^{N_i}\sum_{k=I-i+1}^{I} SIC_{p,i,k}\right) = E\left(N_i\right) Var\left(\sum_{k=I-i+1}^{I} SIC_{p,i,k}\right) + Var\left(N_i\right) E\left(\sum_{k=I-i+1}^{I} SIC_{p,i,k}\right)^2$$

Due to the independence between the $SIC_{p,i,k}$, we have

$$Var\left(\sum_{p=1}^{N_i}\sum_{k=I-i+1}^{I} SIC_{p,i,k}\right) = E\left(N_i\right)\left(\sum_{k=I-i+1}^{I} Var\left(SIC_{p,i,k}\right)\right) + Var\left(N_i\right)\left(\sum_{k=I-i+1}^{I} E\left(SIC_{p,i,k}\right)\right)^2$$

where N_i denotes the future number of claims, which can be estimated with a standard chain ladder method from accident year i (see next section for details).

As we have

$$Var\left(SIC_{p,i,k}\right) = Var\left(SIC_k\right) \text{ for all p and i}$$
$$E\left(SIC_{p,i,k}\right) = E\left(SIC_k\right) \text{ for all p and i}$$

the overall variance for accident year i is

$$Var\left(\sum_{p=1}^{N_i}\sum_{k=I-i+1}^{I} SIC_{p,i,k}\right) = E\left(N_i\right)\left(\sum_{k=I-i+1}^{I} Var\left(SIC_k\right)\right) + Var\left(N_i\right)\left(\sum_{k=I-i+1}^{I} E\left(SIC_k\right)\right)^2 \tag{14}$$

The same can be done for skewness with the law of total skewness and we get

$$SK\left(\sum_{p=1}^{N_i}\sum_{k=I-i+1}^{I} SIC_{p,i,k}\right) = E\left(N_i\right)\left(\sum_{k=I-i+1}^{I} SK\left(SIC_k\right)\right) + SK\left(N_i\right)\left(\sum_{k=I-i+1}^{I} E\left(SIC_k\right)\right)^3 + 3\,Var\left(N_i\right)\left(\sum_{k=I-i+1}^{I} E\left(SIC_k\right)\right) \times \left(\sum_{k=I-i+1}^{I} Var\left(SIC_k\right)\right) \tag{15}$$

6. IBNER BASED ON INDIVIDUAL CLAIMS ANALYSIS

According to Schnieper (1991), the IBNER for accident year i is equal to:

$$IBNER_i = C_{i,n+1-i}\left\{\left[\prod_{j=n+2-i}^{n}\left(1-\delta_j\right)\right]-1\right\} \tag{16}$$

where:

$$\hat{\delta}_j = \frac{\sum_{i=1}^{n+1-j} D_{i,j}}{\sum_{i=1}^{n+1-j} C_{i,j-1}}$$

Table 4. Individual claim triangle characteristics per development year (example)

Claim amount		Development year k			
UWY i	Claim ref	1	2	3	4
1	1	10	12	13	13
	2		8	8	
	3	20	23	24	25
	4		33	34	34
	5	15	17	18	19
2	6		5	6	
	7	40	42	50	
	8	7	9		
3	9	50	52		
	10	5	9		
	11	4			
4	12	8			
	13	12			
Mean			$E(SIC_2)$	$E(SIC_3)$	$E(SIC_4)$
Variance			$Var(SIC_2)$	$Var(SIC_3)$	$Var(SIC_4)$
Skewness			$SK(SIC_2)$	$SK(SIC_3)$	$SK(SIC_4)$

Table 5. Triangle of $D_{i,j}$ with the corresponding $\hat{\delta}_j$

$D_{i,j}$	Development year k			
UWY i	1	2	3	4
1	0	-7	-4	6
2	0	-4	0	
3	0	-6		
4	0			
δ_i	0	-0.11	-0.03	0.06

and $D_{i,j}$ is the decrease in total claims amount between development year j-1 and development year j with respect to claims already known in development year j-1.

In the case of the triangle shown in Table 2, we would have the following resulting triangle of $D_{i,j}$ with the corresponding $\hat{\delta}_j$, shown in Table 5.

In Schnieper (1991), we also have that:

$$Var\left(D_{i,j}\right) = C_{i,j-1}\tau_j^2$$

where

$$\widehat{\tau}_j^2 = \frac{1}{n-j}\sum_{i=1}^{n+1-j}\frac{\left(D_{i,j}-\hat{\delta}_j C_{i,j-1}\right)^2}{C_{i,j-1}}$$

By natural extension, the skewness estimate can be derived as follows:

$$SK\left(D_{i,j}\right) = C_{i,j-1}^{\frac{3}{2}}\zeta_j^3$$

where

$$\hat{\zeta}_j^3 = \frac{1}{n-j}\sum_{i=1}^{n+1-j}\frac{\left(D_{i,j}-\hat{\delta}_j C_{i,j-1}\right)^3}{C_{i,j-1}^{\frac{3}{2}}}$$

As in Schnieper (1991), let's denote $\hat{\theta}_i = (\delta_{n+2-i},\ldots,\delta_n)$. Developing $IBNER_i\left(\hat{\theta}_i\right)$ in a Taylor series, we obtain

$$IBNER_i\left(\hat{\theta}_i\right) = IBNER_i\left(\theta_i\right) + \sum_{j=n+2-i}^{n}\frac{\delta\, IBNER_i\left(\delta_j\right)}{\delta\,\delta_j}$$

Due to the independence of the $D_{i,j}$, we can calculate the mean standard error (hereinafter "mse") as follows (Schnieper 1991):

$$mse\left(IBNER_i\left(\hat{\theta}_i\right)\right) = E\left(IBNER_i\left(\hat{\theta}_i\right) - IBNER_i\left(\theta_i\right)\right)^2 = \sum_{j=n+2-i}^{n}\left(\frac{\delta\, IBNER_i\left(\delta_j\right)}{\delta\,\delta_j}|_{\theta_i=\hat{\theta}_i}\right)^2 Var\left(\hat{\delta}_j\right)$$

where

$$Var\left(\hat{\delta}_j\right) = \frac{\tau_j^2}{\sum_{i=1}^{n+1-j}C_{i,j-1}}$$

Following equation (16), we have:

$$\frac{\delta\, IBNER_i\left(\delta_j\right)}{\delta\,\delta_j}|_{\theta_i=\hat{\theta}_i} = -C_{i,n+1-i}\prod_{\substack{k=n+2-i\\k\neq j}}^{n}\left(1-\delta_k\right)$$

which leads to:

$$\begin{aligned}&\operatorname{mse}\left(\text{IBNER}_i\left(\hat{\theta}_i\right)\right)\\&=C_{i,n+1-i}^2\sum_{j=n+2-i}^{n}\left(\prod_{k=n+2-i}^{n}(1-\delta_k)\right)^2\frac{\operatorname{Var}\left(\hat{\delta}_J\right)}{(1-\delta_j)^2}\end{aligned}\tag{17}$$

The same can be done for the skewness estimation which leads to the following formula:

$$\begin{aligned}&SK\left(\text{IBNER}_i\left(\hat{\theta}_i\right)\right)\\&=-C_{i,n+1-i}^3\sum_{j=n+2-i}^{n}\left(\prod_{k=n+2-i}^{n}(1-\delta_k)\right)^3\frac{SK\left(\hat{\delta}_j\right)}{(1-\delta_j)^3}\end{aligned}\tag{18}$$

where

$$SK\left(\hat{\delta}_j\right)=\frac{\zeta_j^3}{\sum_{i=1}^{n+1-j}C_{i,j-1}^{\frac{3}{2}}}$$

7. ESTIMATION OF OVERALL SKEWNESS AND STANDARD DEVIATION

After estimating IBNYR and IBNER by UWY, based on Mack (1993), the overall standard deviation can easily be calculated as per the formula below. Let's denote s.d.(R_i), the standard deviation of the reserve of UWY i (see 1.b for definition of (R_i) and s.d.(R) the standard deviation overall all UWYs ($R=R_2+\ldots+R_n$). Then:

$$\begin{aligned}&s.d.\left(\widehat{R}\right)^2\\&=\sum_{i=2}^{I}\left\{s.d.\left(\widehat{R}_i\right)^2+\widehat{C}_{iI}^{CL}\left(\sum_{j=i+1}^{I}\widehat{C}_{jI}^{CL}\right)\sum_{k=I+1-i}^{I-1}\frac{2\,\frac{\widehat{\sigma}_k^2}{\widehat{f}_k^2}}{\sum_{n=1}^{I-k}C_{n,k}}\right\}\end{aligned}\tag{19}$$

Let's denote the following correlations:

$$r_{i,j}=\frac{\widehat{C}_{iI}^{CL}\widehat{C}_{jI}^{CL}\sum_{k=I+1-i}^{I-1}\frac{\frac{\widehat{\sigma}_k^2}{\widehat{f}_k^2}}{\sum_{n=1}^{I-k}C_{n,k}}}{\sqrt{Var\left(\widehat{C}_{iI}^{CL}\right)Var\left(\widehat{C}_{jI}^{CL}\right)}}$$

And as $Var\left(\widehat{C}_{iI}^{CL}\right)=s.d.\left(\widehat{R}_i\right)^2$, we can write equation (19) in the following form:

$$\text{s.d.}\left(\widehat{R}\right)^2=\begin{pmatrix}s\cdot d\cdot\left(\widehat{R_1}\right)\\\ldots\\\text{s.d.}\left(\widehat{R_1}\right)\end{pmatrix}\begin{pmatrix}1&r_{12}&\ldots\\r_{12}&\ldots&\ldots\\\ldots&\ldots&1\end{pmatrix}\begin{pmatrix}\text{s.d.}\left(\widehat{R_1}\right)\\\ldots\\\text{s.d.}\left(\widehat{R_1}\right)\end{pmatrix}$$

In order to aggregate the standard deviation of the proposed method based on IBNYR and IBNER by UWY, we can use the same correlation matrix on the standard deviations of each UWY to get the overall standard deviation.

As for the overall skewness, before giving a general formula, we are going to limit the estimation to the case of three accident years X1, X2, X3 as shown in the framework of lemma 1 (see appendix B). The question is the estimation of $SK(X_1+X_2+X_3)$. It develops as follows:

$$\begin{aligned}SK(X_1+X_2+X_3)&=E\left[\left((X_1+X_2+X_3)-E(X_1+X_2+X_3)\right)^3\right]\\&=E\left[(X_1+X_2+X_3)^3\right]\\&\quad-3E\left[(X_1+X_2+X_3)^2\right]E\left[(X_1+X_2+X_3)\right]\\&\quad+2E[(X_1+X_2+X_3)]^3\end{aligned}$$

$$\begin{aligned}SK(X_1+X_2+X_3)=&\,SK[X_1]+SK[X_2]+SK[X_3]\\&+3E[X_1X_2(X_1+X_2)]\\&\quad-6E[X_1X_2](E[X_1]+E[X_2])\\&+3E[X_1X_3(X_1+X_3)]\\&\quad-6E[X_1X_3](E[X_1]+E[X_3])\\&+3E[X_2X_3(X_2+X_3)]\\&\quad-6E[X_2X_3](E[X_2]+E[X_3])\\&+3E[X_1]\left[E(X_2)^2-\operatorname{Var}(X_2)\right]\\&+3E[X_1]\left[E(X_3)^2-\operatorname{Var}(X_3)\right]\\&+3E[X_2]\left[E(X_1)^2-\operatorname{Var}(X_1)\right]\\&+3E[X_2]\left[E(X_3)^2-\operatorname{Var}(X_3)\right]\\&+3E[X_3]\left[E(X_1)^2-\operatorname{Var}(X_1)\right]\\&+3E[X_3]\left[E(X_2)^2-\operatorname{Var}(X_2)\right]\\&+6E[X_1X_2X_3]+12E[X_1]E[X_2]E[X_3]\\&-6E[X_1X_2]E[X_3]\\&\quad-6E[X_1X_3]E[X_2]-6E[X_2X_3]E[X_1]\end{aligned}$$

Following lemma 1 (see appendix B), we have

$$\begin{aligned}E\left[X_1^2X_2\right]&=\operatorname{Cov}\left(X_1^2,X_2\right)+E\left(X_1^2\right)E(X_2)\\&=E(X_1)\left(1+\frac{\operatorname{Var}(X_1)}{E(X_1)^2}\right)\operatorname{Cov}(X_1,X_2)\left(2+\frac{\operatorname{Cov}(X_1,X_2)}{E(X_1)E(X_2)}\right)\\&\quad+\left(\operatorname{Var}(X_1)+E(X_1)^2\right)E(X_2)\end{aligned}$$

For the following elements of the above equation, we therefore get

$$\begin{aligned}&E[X_1X_2(X_1+X_2)]-2E[X_1X_2](E[X_1]+E[X_2])\\&+E[X_1]\left[E(X_2)^2-\operatorname{Var}(X_2)\right]+E[X_2]\left[E(X_1)^2-\operatorname{Var}(X_1)\right]\\&=E(X_1)\left(1+\frac{\operatorname{Var}(X_1)}{E(X_1)^2}\right)\operatorname{Cov}(X_1,X_2)\left(2+\frac{\operatorname{Cov}(X_1,X_2)}{E(X_1)E(X_2)}\right)\\&\quad+\left(\operatorname{Var}(X_1)+E(X_1)^2\right)E(X_2)\\&+E(X_2)\left(1+\frac{\operatorname{Var}(X_2)}{E(X_2)^2}\right)\operatorname{Cov}(X_1,X_2)\left(2+\frac{\operatorname{Cov}(X_1,X_2)}{E(X_1)E(X_2)}\right)\\&\quad+\left(\operatorname{Var}(X_2)+E(X_2)^2\right)E(X_1)\\&-2(E[X_1]+E[X_2])(\operatorname{Cov}(X_1,X_2)+E(X_1)E(X_2))\\&+E[X_1]\left[E(X_2)^2-\operatorname{Var}(X_2)\right]+E[X_2]\left[E(X_1)^2-\operatorname{Var}(X_1)\right]\\&=\operatorname{Cov}(X_1,X_2)\left(2+\frac{\operatorname{Cov}(X_1,X_2)}{E(X_1)E(X_2)}\right)\\&\quad\times\left[E(X_1)\left(1+\frac{\operatorname{Var}(X_1)}{E(X_1)^2}\right)+E(X_2)\left(1+\frac{\operatorname{Var}(X_2)}{E(X_2)^2}\right)\right]\\&-2(E[X_1]+E[X_2])\operatorname{Cov}(X_1,X_2)\\&=\operatorname{Cov}(X_1,X_2)\left(2+\frac{\operatorname{Cov}(X_1,X_2)}{E(X_1)E(X_2)}\right)\left[\frac{\operatorname{Var}(X_1)}{E(X_1)}+\frac{\operatorname{Var}(X_2)}{E(X_2)}\right]\\&\quad+\operatorname{Cov}(X_1,X_2)^2\left[\frac{E[X_1]+E[X_2]}{E[X_1]E[X_2]}\right]\end{aligned}$$

In addition, using lemma 2 (see appendix B), we get

$$\begin{aligned}&E[X_1X_2X_3]+2E[X_1]E[X_2]E[X_3]\\&\quad-E[X_1X_2]E[X_3]-E[X_1X_3]E[X_2]\\&\quad-E[X_2X_3]E[X_1]\\&=E[X_1]E[X_2]E[X_3]\left(2+\left(1+\frac{\operatorname{Cov}(X_1,X_2)}{E[X_1]E[X_2]}\right)\right.\\&\quad\left.\times\left(1+\frac{\operatorname{Cov}(X_1,X_3)}{E[X_1]E[X_3]}\right)\times\left(1+\frac{\operatorname{Cov}(X_2,X_3)}{E[X_2]E[X_3]}\right)\right)\\&\quad-E[X_3]\operatorname{Cov}(X_1,X_2)-E[X_1]E[X_2]E[X_3]\\&\quad-E[X_2]\operatorname{Cov}(X_1,X_3)-E[X_1]E[X_2]E[X_3]\\&\quad-E[X_1]\operatorname{Cov}(X_2,X_3)-E[X_1]E[X_2]E[X_3]\\&=r_{12}r_{13}r_{23}\sqrt{\operatorname{Var}(X_1)\operatorname{Var}(X_2)\operatorname{Var}(X_3)}\\&\quad\times\left(\frac{\sqrt{\operatorname{Var}(X_1)\operatorname{Var}(X_2)\operatorname{Var}(X_3)}}{E[X_1]E[X_2]E[X_3]}+\frac{\sqrt{\operatorname{Var}(X_1)}}{r_{23}E[X_1]}+\frac{\sqrt{\operatorname{Var}(X_2)}}{r_{13}E[X_2]}+\frac{\sqrt{\operatorname{Var}(X_3)}}{r_{12}E[X_3]}\right)\end{aligned}$$

In the case of three accident years and under the restrictions indicated in lemmas 1 and 2, we therefore find the following aggregate skewness:

Table 6. Individual claims triangles – Incurred statistics

UWY	Known claims	Known claims incurred	Devlpt year	Stdev(SIC)	E(SIC)	SK(SIC)
2007	49	11'855'386	10	1'597	273	1.97E+10
2008	28	1'608'242	9	2'441	423	8.93E+10
2009	31	8'030'481	8	59'555	0	-4.01E+13
2010	40	27'873'128	7	408'682	24'941	6.66E+17
2011	41	4'186'378	6	103'044	2'740	-1.15E+15
2012	45	4'742'007	5	531'210	42'249	2.04E+18
2013	20	5'478'548	4	177'257	16'023	2.83E+16
2014	20	14'857'473	3	332'358	54'386	3.33E+17
2015	6	9'172'509	2	970'352	190'389	5.74E+18
2016	2	213'825	1	732'010	187'448	1.85E+18

$$\begin{aligned}SK(X_1+X_2+X_3) = {} & SK[X_1]+SK[X_2]+SK[X_3] \\ & +3r_{12}\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)} \\ & \times\left[\frac{\mathrm{Var}(X_1)}{E(X_1)}+\frac{\mathrm{Var}(X_2)}{E(X_2)}\right]\left[2+r_{12}\frac{\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)}}{E[X_1]\,E[X_2]}\right] \\ & +3r_{12}^2\,\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)\left[\frac{E[X_1]+E[X_2]}{E[X_1]\,E[X_2]}\right] \\ & +3r_{13}\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_3)} \\ & \times\left[\frac{\mathrm{Var}(X_1)}{E(X_1)}+\frac{\mathrm{Var}(X_3)}{E(X_3)}\right]\left[2+r_{13}\frac{\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_3)}}{E[X_1]\,E[X_3]}\right] \\ & +3r_{13}^2\,\mathrm{Var}(X_1)\,\mathrm{Var}(X_3)\left[\frac{E[X_1]+E[X_3]}{E[X_1]\,E[X_3]}\right] \\ & +3r_{23}\sqrt{\mathrm{Var}(X_2)\,\mathrm{Var}(X_3)} \\ & \times\left[\frac{\mathrm{Var}(X_2)}{E(X_2)}+\frac{\mathrm{Var}(X_3)}{E(X_3)}\right]\left[2+r_{23}\frac{\sqrt{\mathrm{Var}(X_2)\,\mathrm{Var}(X_3)}}{E[X_2]\,E[X_3]}\right] \\ & +3r_{23}^2\,\mathrm{Var}(X_2)\,\mathrm{Var}(X_3) \\ & \times\left[\frac{E[X_2]+E[X_3]}{E[X_2]\,E[X_3]}\right] \\ & +6r_{12}r_{13}r_{23}\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)\,\mathrm{Var}(X_3)} \\ & \times\left(\frac{\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)\,\mathrm{Var}(X_3)}}{E[X_1]\,E[X_2]\,E[X_3]}\right. \\ & \left.+\frac{\sqrt{\mathrm{Var}(X_1)}}{r_{23}E[X_1]}+\frac{\sqrt{\mathrm{Var}(X_2)}}{r_{13}E[X_2]}+\frac{\sqrt{\mathrm{Var}(X_3)}}{r_{12}E[X_3]}\right)\end{aligned}$$

The generalization of the above equation to more than three accident years is provided in appendix B.

8. NUMERICAL EXAMPLES

The formulae above are applied to a set of individual claims provided on the link

[put the link on Variance Journal additional docs]

Table 6 shows some sample statistics related to these individual claims.

The columns in Table 6 are:

- Known claims: These are the cumulative number of claims known at the end of 2016 for each UWY;
- Known claims incurred: These are the cumulative incurred amounts for the known claims at the end of 2016 for each UWY;
- Stdev(SIC): Represents the standard deviation of the incremental incurred for each development year;
- E(SIC): Represents the average incremental incurred for each development year;
- SK(SIC): Represents the skewness of the incremental incurred for each development year.

As mentioned in section 4, the first step is to estimate the future number of claims and the volatility and skewness of the distribution of the future number of claims. This is done on the sheet "Triangle incurred chain ladder" of the example that uses the Excel macro "Mack1999" (relating to the article "An Approximation of the Nonlife Reserve Risk Distribution Using the Cornish-Fisher Expansion" (Dal Moro 2013)).

The results of the calculation are presented in Table 7. where

- Future claims represent the number of future claims estimated by the chain ladder model applied to the cumulative number of claims per UWY;
- CoV(N): Corresponds to the coefficient of variation of the number of future claims estimated according to Mack (1993);
- SK(N): Corresponds to the skewness of the number of future claims according to Dal Moro (2013).

As a next step, it is necessary to estimate the values of 1-δ_i , Var(δ_i) and SK(δ_i) as in Schnieper (1991) and in section 6 above. The results are provided in Table 8.

The IBNER can be calculated according to equation (16) and the IBNYR can be calculated as

$$IBNYR_i = Nb\ Future\ claims_i \times \sum_{j=n-i+1}^{n} E\left(SIC_j\right)$$

And the overall IBNR reserves is the sum of IBNER and IBNYR as shown in Table 9. A comparison to the IBNR reserves provided by the simple application of the chain ladder method to the incurred triangle is also provided.

One of the main differences is on UWY 2015 as the known incurred claim of 9,172,509 is certainly very high and represents an outlier. Projected with chain ladder, it provides a very high required IBNR. The same applies also to UWY 2014 where the incurred amount of 14,857,473 seems to be an outlier. It must be borne in mind that a reserving actuary would correct these two figures on taking out the large before projecting with chain ladder. Alternatively, a Bornhuetter-Ferguson or a Cape Cod method would likely be applied.

Having estimated the overall IBNR reserves, we can now estimate the standard deviation of these reserves. As IB-

Table 7. Individual claims triangle – Number of claims statistics

UWY	Known claims	Known claims incurred	Future claims $E(N_i)$	$CoV(N_i)$	$SK(N_i)$
2007	49	11'855'386	2.0	45%	0.0
2008	28	1'608'242	1.6	51%	0.0
2009	31	8'030'481	2.1	48%	0.0
2010	40	27'873'128	4.4	37%	0.5
2011	41	4'186'378	6.8	69%	42.9
2012	45	4'742'007	9.4	64%	68.0
2013	20	5'478'548	7.7	70%	58.9
2014	20	14'857'473	16.4	80%	2689.8
2015	6	9'172'509	11.2	100%	1952.8
2016	2	213'825	31.3	78%	15528.5

Table 8. Values of 1-δ_i , Var(δ_i) and SK(δ_i)

Devlpt year	1-δ_i	Var(δ_i)	SK(δ_i)
10	1,00	-	0,00
9	1,00	0,00	0,00
8	0,96	0,00	0,00
7	1,16	0,09	-0,04
6	0,99	0,00	0,00
5	0,99	0,00	-0,00
4	1,05	0,01	-0,00
3	1,11	0,00	-0,00
2	2,09	0,85	-2,65
1	6,89	9,76	-89,24

Table 9. Individual claims triangle – Reserves estimation

UWY	Known claims incurred	IBNER	Future claims	E(SIC)	Reserves (indiv based)	Reserves (chain ladder)
2007	11'855'386	-7'293	2.0	272.6	-6'737	47'058
2008	1'608'242	638	1.6	422.9	1'731	10'065
2009	8'030'481	-340'935	2.1	0.0	-339'495	-92'338
2010	27'873'128	3'044'617	4.4	24940.8	3'156'878	4'099'823
2011	4'186'378	388'768	6.8	2740.4	581'909	660'921
2012	4'742'007	396'841	9.4	42248.8	1'060'251	1'800'448
2013	5'478'548	783'004	7.7	16022.7	1'449'799	2'706'627
2014	14'857'473	4'021'821	16.4	54385.6	6'333'436	14'564'159
2015	9'172'509	15'201'827	11.2	190389.1	18'907'496	50'258'563
2016	213'825	3'700'157	31.3	187447.7	19'951'655	27'682'338
				Total	51'096'924	101'737'665

NYR and IBNER are independent and based on equations (14) and (17), we have the following overall standard deviation by UWY for the IBNR reserves:

Table 10. Standard deviation comparison

UWY	Standard deviation (indiv based)	Standard deviation (chain ladder)
2007	2’295	227
2008	3’699	2’461
2009	85’809	500’929
2010	1’213’251	10’913’341
2011	1’644’767	3’158’149
2012	2’531’774	3’974’048
2013	2’557’232	4’983’313
2014	5’825’307	14’516’104
2015	6’245’286	50’320’626
2016	15’098’430	51’778’916
Total	21’484’301	89’931’007

$$\begin{aligned}\operatorname{Var}(IBNR_i) &= \operatorname{Var}(IBNER_i) + \operatorname{Var}(IBNYR_i)\\ &= C_{i,n+1-i}^2 \sum_{j=n+2-i}^{n} \left(\prod_{k=n+2-i}^{n}(1-\delta_k)\right)^2 \frac{\operatorname{Var}\left(\hat{\delta}_j\right)}{(1-\delta_j)^2}\\ &\quad + E(N_i)\left(\sum_{k=n-i+1}^{n}\operatorname{Var}(SIC_k)\right)\\ &\quad + \operatorname{Var}(N_i)\left(\sum_{k=n-i+1}^{n}E(SIC_k)\right)^2\end{aligned}$$

The resulting standard deviations using that equation is compared to the chain ladder standard deviation (Mack 1993) in Table 10.

As for the reserve estimation, there are significant differences between the chain-ladder standard deviation and the standard deviation based on this method for UWY 2010, 2014, 2015 and 2016. For the latter years, the same reasons as for the reserve estimation should explain the differences: There seems to be outliers in the data on the most recent developments. As for UWY 2010, the significant increases in development N+2, N+3 and N+5 are due to large losses: The Mack standard deviation is therefore influenced by these outliers.

Finally, due to the independence of IBNER and IBNYR, we can calculate the skewness for the different UWYs according to equations (15) and (18) as shown below:

$$\begin{aligned}SK(IBNR_i) &= E(N_i)\left(\sum_{k=n-i+1}^{n} SK(SIC_k)\right)\\ &\quad + SK(N_i)\left(\sum_{k=n-i+1}^{n}E(SIC_k)\right)^3\\ &\quad + 3\operatorname{Var}(N_i)\left(\sum_{k=n-i+1}^{n}E(SIC_k)\right)\left(\sum_{k=n-i+1}^{n}\operatorname{Var}(SIC_k)\right)\\ &\quad - C_{i,n+1-i}^3\sum_{j=n+2-i}^{n}\left(\prod_{k=n+2-i}^{n}(1-\delta_k)\right)^3\frac{SK\left(\hat{\delta}_J\right)}{(1-\delta_j)^3}\end{aligned}$$

The resulting skewness using the above equation is compared to the chain ladder skewness (Dal Moro 2013) in Table 11.

Overall, the skewness coefficients are relatively comparable except for UWY 2016 where the chain ladder skewness is much higher. As for the standard deviation and the reserve estimation, it is due to the outliers present in the aggregated triangle.

9. CONCLUSION

This paper is a first attempt to unify all the usual reserving methods into one overarching method based on an analysis of individual claims. This paper also provides an overall standard deviation and an overall skewness for all UWYs.

The proposed reserving methodology is easy to implement based on individual claims and is more stable than the chain ladder, Bornhuetter-Ferguson or Cape Cod method. The underlying rationale for the unified reserving method relies on the fact that the volatility of a line of business always depends on the confidence in the incurred/payment pattern. When the incurred/payment pattern is uncertain, the line of business is considered volatile. On the other hand, when the incurred/payment pattern is stable across the accident or underwriting years, the line of business is considered stable. Based on this ascertainment, the proposed methodology looks into the information that the individual claim can provide and derives an estimate for reserves, standard deviation and skewness. A numerical example is proposed with all the detailed calculation being available.

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Table 11. Skewness coefficients comparison

UWY	Skewness (indiv based)	Skewness coef (indiv based)	Skewness coef (chain ladder)
2007	4.20E+10	3.5	0.00
2008	1.83E+11	3.6	-0.54
2009	-7.54E+13	-0.1	-0.62
2010	2.95E+18	1.7	0.42
2011	4.86E+18	1.1	0.71
2012	2.89E+19	1.8	0.69
2013	2.49E+19	1.5	0.78
2014	1.02E+20	0.5	0.69
2015	3.61E+20	1.5	0.96
2016	4.46E+21	1.3	2.12
Total	8.94E+21	0.90	1.75

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APPENDICES

APPENDIX A — DESCRIPTION OF THE ATTACHED EXCEL WORKBOOK TO THIS ARTICLE

In order to make the application of this article easier for the reader, an Excel workbook is provided. This workbook contains the following sheets:

- ReadMe: This sheet provides a description of the way in which the two macros "Mack1999" and "OverallSkew1" should be used.
  - The macro "Mack1999" is applying the chain ladder projection method to a triangle. In addition to the ultimate claims reserves, the Mack volatility (Mack 1993) and the skewness of the resulting reserves (Dal Moro 2013) are also provided.
  - The macro "OverallSkew1" calculates the overall skewness of the proposed unified method based on the skewness estimates of each UWYs.
- Data: This sheet contains the individual claims data used in this article. On the line 326 to 329, different indicators are also estimated (mean, standard deviation, skewness and δ_i from Schnieper (1991)) are also provided.
- Triangle Di Schnieper: This sheet provides the application of the IBNER estimates given in Schnieper (1991).
- Triangle Incurred Chain Ladder: This sheet shows the result of the application of the macro Mack1999 on the triangle of incurred corresponding to the aggregation of individual claims data in a standard UWY x Development year triangle.
- Triangle nb UWY Reporting dvlpt: This sheet shows the result of the application of the macro Mack1999 on the triangle of number of claims corresponding to the aggregation of individual claims data in a standard number of claims triangle by UWY x Development year.
- Summary: Provides the results of the proposed reserving method provided in this article. The overall skewness is calculated on running the macro "OverallSkew1" (see sheet "ReadMe" for details).

APPENDIX B – FORMULAE FOR AGGREGATING SKEWNESS ACROSS UWYS

Lemma 1

Assume the vector $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ follows a bivariate Gaussian distribution

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & r_{12}\sigma_1\sigma_2 \\ r_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right)$$

where r_{12} denotes the correlation between Y_1 and Y_2

$$r_{12} = \frac{E(Y_1 Y_2) - E(Y_1)E(Y_2)}{\sqrt{Var(Y_1)\,Var(Y_2)}} = \frac{Cov(Y_1, Y_2)}{\sqrt{Var(Y_1)\,Var(Y_2)}}$$

If we denote

$$X_1 = \exp(Y_1)$$
$$X_2 = \exp(Y_2)$$

then we have

$$Cov\left(X_1^2, X_2\right) = E(X_1)\left(1 + \frac{Var(X_1)}{E(X_1)^2}\right) Cov(X_1, X_2)\left(2 + \frac{Cov(X_1, X_2)}{E(X_1)\,E(X_2)}\right)$$

Proof:

Following the assumptions of the lemma, we have the following properties (Embrechts, Frey, and McNeil 2005):

$$\forall i \in [1,2] \quad E(X_i) = \exp\left(\mu_i + \frac{\sigma_i^2}{2}\right)$$
$$\mathrm{Var}(X_i) = E(X_i)\left[\exp\left(\sigma_i^2\right) - 1\right]$$
$$\mathrm{Cov}(X_1, X_2) = E(X_1)\,E(X_2)\left[\exp(r_{12}\sigma_1\sigma_2) - 1\right]$$

If we denote

$$\widetilde{X} = \exp\left(\widetilde{Y}\right)$$

$$\text{with } \widetilde{\mathrm{Y}} = \begin{pmatrix} 2Y_1 \\ Y_2 \end{pmatrix} \sim N\left[\begin{pmatrix} 2\mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} 4\sigma_1^2 & 2r_{12}\sigma_1\sigma_2 \\ 2r_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right]$$

then we have

$$\begin{aligned} \mathrm{Cov}\left(X_1^2, X_2\right) &= \mathrm{Cov}\left(\tilde{X}_1, \tilde{X}_2\right) \\ &= \exp\left(2\mu_1 + \mu_2 + \frac{1}{2}\left(4\sigma_1^2 + \sigma_2^2\right)\right)\left(\exp(2r_{12}\sigma_1\sigma_2) - 1\right) \\ &= E\left(X_1^2\right)E(X_2)\left(\exp(2r_{12}\sigma_1\sigma_2) - 1\right) \\ &= \frac{E\left(X_1^2\right)}{E(X_1)}\frac{\left(\exp(2r_{12}\sigma_1\sigma_2) - 1\right)}{\left(\exp(r_{12}\sigma_1\sigma_2) - 1\right)}\mathrm{Cov}(X_1, X_2) \end{aligned}$$

As we have

$$\begin{aligned} \forall a \frac{a^2 - 1}{a - 1} &= 1 + a\frac{\left(\exp(2r_{12}\sigma_1\sigma_2) - 1\right)}{\left(\exp(r_{12}\sigma_1\sigma_2) - 1\right)} \\ &= 1 + \exp(r_{12}\sigma_1\sigma_2) \\ &= 2 + \frac{\mathrm{Cov}(X_1, X_2)}{E(X_1)\,E(X_2)} \end{aligned}$$

and

$$\begin{aligned} \frac{E\left(X_1^2\right)}{E(X_1)} &= \frac{\exp\left(2\mu_1 + 2\sigma_1^2\right)}{\exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right)} \\ &= E(X_1)\left(1 + \frac{Var(X_1)}{E(X_1)^2}\right) \end{aligned}$$

we get to the conclusion that

$$\begin{aligned} & Cov\left(X_1^2, X_2\right) \\ &= E(X_1)\left(1 + \frac{Var(X_1)}{E(X_1)^2}\right) Cov(X_1, X_2)\left(2 + \frac{Cov(X_1, X_2)}{E(X_1)\,E(X_2)}\right) \end{aligned}$$

Lemma 2

Assume Y_1, Y_2 and Y_3 are three random variables following a multivariate Gaussian distribution

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & r_{12}\sigma_1\sigma_2 & r_{13}\sigma_1\sigma_3 \\ r_{12}\sigma_1\sigma_2 & \sigma_2^2 & r_{23}\sigma_2\sigma_3 \\ r_{13}\sigma_1\sigma_3 & r_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix} \right)$$

where r_{ij} denotes the correlation between Y_i and Y_j

$$r_{ij} = \frac{Cov(Y_i, Y_j)}{\sqrt{Var(Y_i)\,Var(Y_j)}}$$

If we denote

$$X_1 = \exp(Y_1) X_2 = \exp(Y_2) X_3 = \exp(Y_3)$$

then we have

$$\begin{aligned} & E[X_1X_2X_3] = E(X_1)\,E(X_2)\,E(X_3) \\ & \quad \times \left(1 + \frac{Cov(X_1, X_2)}{E(X_1)\,E(X_2)}\right)\left(1 + \frac{Cov(X_1, X_3)}{E(X_1)\,E(X_3)}\right)\left(1 + \frac{Cov(X_2, X_3)}{E(X_2)\,E(X_3)}\right) \end{aligned}$$

Proof:

Following the assumptions of the lemma, we have the following properties (Embrechts, Frey, and McNeil 2005):

$$\begin{aligned} E[X_1X_2X_3] &= \exp\left\{\mu_1 + \mu_2 + \mu_3 + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)\right. \\ & \quad \left. + r_{12}\sigma_1\sigma_2 + r_{13}\sigma_1\sigma_3 + r_{23}\sigma_2\sigma_3\right\} \\ &= E[X_1]\,E[X_2]\,E[X_3]\exp\left\{r_{12}\sigma_1\sigma_2 + r_{13}\sigma_1\sigma_3 + r_{23}\sigma_2\sigma_3\right\} \\ &= E[X_1]\,E[X_2]\,E[X_3] \\ & \quad \times \left(1 + \frac{\mathrm{Cov}(X_1, X_2)}{E[X_1]\,E[X_2]}\right) \\ & \quad \times \left(1 + \frac{\mathrm{Cov}(X_1, X_3)}{E[X_1]\,E[X_3]}\right) \\ & \quad \times \left(1 + \frac{\mathrm{Cov}(X_2, X_3)}{E[X_2]\,E[X_3]}\right) \end{aligned}$$

which ends the proof.

The generalization of the formula determined for three accident years is easily done. The result is

$$\begin{aligned} SK\left(\sum_{i=2}^{I}\hat{C}_{i,I}\right) &= \sum_{i=2}^{I} SK\left(\hat{C}_{i,I}\right) \\ & + 3\sum_{i=2}^{I-1}\sum_{j=i+1}^{I} r_{ij}\sqrt{\mathrm{Var}\left(\hat{C}_{i,I}\right)\mathrm{Var}\left(\hat{C}_{j,I}\right)} \\ & \times \left[\frac{\mathrm{Var}\left(\hat{C}_{i,I}\right)}{E\left(\hat{C}_{i,I}\right)} + \frac{\mathrm{Var}\left(\hat{C}_{j,I}\right)}{E\left(\hat{C}_{j,I}\right)}\right] \\ & \times \left[2 + r_{ij}\frac{\sqrt{\mathrm{Var}\left(\hat{C}_{i,I}\right)\mathrm{Var}\left(\hat{C}_{j,I}\right)}}{E\left[\hat{C}_{i,I}\right]E\left[\hat{C}_{j,I}\right]}\right] \\ & + 3\sum_{i=2}^{\mathrm{I}-1}\sum_{j=i+1}^{I} r_{ij}^2\,\mathrm{Var}\left(\hat{C}_{i,I}\right)\mathrm{Var}\left(\hat{C}_{j,I}\right) \\ & \times \left[\frac{E\left[\hat{C}_{i,I}\right] + E\left[\hat{C}_{j,I}\right]}{E\left[\hat{C}_{i,I}\right]E\left[\hat{C}_{j,I}\right]}\right] \\ & + 6\sum_{i=2}^{I-2}\sum_{j=i+1}^{I-1}\sum_{k=j+1}^{I} r_{ij}r_{ik}r_{jk}\sqrt{\mathrm{Var}\left(\hat{C}_{i,I}\right)\mathrm{Var}\left(\hat{C}_{j,I}\right)\mathrm{Var}\left(\hat{C}_{k,I}\right)} \\ & \times \left(\frac{\sqrt{\mathrm{Var}\left(\hat{C}_{i,I}\right)\mathrm{Var}\left(\hat{C}_{j,I}\right)\mathrm{Var}\left(\hat{C}_{k,I}\right)}}{\frac{E\left[\hat{C}_{i,I}\right]E\left[\hat{C}_{j,I}\right]E\left[\hat{C}_{k,I}\right]}{\sqrt{\mathrm{Var}\left(\hat{C}_{i,I}\right)}} + \frac{\sqrt{\mathrm{Var}\left(\hat{C}_{j,I}\right)}}{r_{jk}E\left[\hat{C}_{i,I}\right]} + \frac{\sqrt{\mathrm{Var}\left(\hat{C}_{k,I}\right)}}{r_{ij}E\left[\hat{C}_{k,I}\right]}}\right) \end{aligned} \tag{21}$$

where

$E\left[\widehat{C}_{i,I}\right]$, $Var\left(\widehat{C}_{i,I}\right)$, r_{ij} are given by (Mack 1993)

$$E\left[\hat{C}_{i,I}\right] = C_{i,I+1-i}\hat{f}_{I+1-i}\ldots\hat{f}_{I-1}$$

$$\mathrm{Var}\left(\hat{C}_{i,I}\right) = \hat{C}_{i,I}^2\sum_{k=I+1-i}^{I-1}\frac{\hat{\sigma}_k^2}{\hat{f}_k}\left(\frac{1}{\hat{C}_{i,k}} + \frac{1}{\sum_{j=1}^{I-k}C_{j,k}}\right)$$

$$r_{i,j} = \frac{\hat{C}_{i,I}^2\hat{C}_{j,I}\sum_{k=I+1-i}^{I-1}\frac{\hat{f}_k^2}{\sum_{n=1}^{I-k}C_{n,k}}}{\sqrt{\mathrm{Var}\left(\hat{C}_{i,I}\right)\mathrm{Var}\left(\hat{C}_{j,I}\right)}}$$

APPENDIX C — TEST OF THE PROJECTION METHOD

In order to test the projection method, the model is fitted to all but the last one diagonal of the individual claims data. Then the last diagonal is projected and compared to the real data as shown in Table A.1.

Even though any conclusion on such limited data is difficult to draw, here are few ascertainments:

- For UWY 2007, 2008, 2010, 2012, 2013, 2014 and 2016, the prediction is relatively close to the real data. For these UWYs, there do not seem to be any outliers in the data;
- For UWY 2009 and 2015, the real data clearly shows an outlier whereby there is a significant and unexpected jump of the incurred;
- Finally, for UWY 2011, the real data shows an outlier whereby the incurred stays constant while it should increase.

Overall, when eliminating the outlier, the proposed projection method seems to fit well the real data.

Table A.1. Individual claims triangle converted to a standard projection triangle

UWY	N	N+1	N+2	N+3	N+4	N+5	N+6	N+7	N+8	N+9	N+10	Last diagonal (real)	Diagonal prediction
2006	161295	541538	1463061	1659941	1716881	1709176	1694521	1710572	1715091	1716765	1723579	1723579	
2007	8189	844617	8445206	11710555	10948955	10945110	11696666	11809172	11826189	11855386		11’855’386	11’831’069
2008	100001	1520381	2029839	1958706	2029410	2232183	3207024	1901910	1608242			1’608’242	1’865’609
2009	102001	1321074	2917221	2988582	3107580	4083554	3611623	8030481				8’030’481	3’026’585
2010	202741	888362	16369437	20190577	21526661	28665051	27873128					27’873’128	29’116’771
2011	30001	423676	1332416	2076608	4160813	4186378						4’186’378	6’305’163
2012	72057	230749	826077	3955426	4742007							4’742’007	4’873’290
2013	344785	3483700	4354424	5478548								5’478’548	5’928’896
2014	32799	6820576	14857473									14’857’473	17’532’480
2015	200001	9172509										9’172’509	1’440’591
2016	213825											213’825	113’988