Multivariate Copulas for Financial Modeling

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ABSTRACT

Although the copula literature has many instances of bivariate copulas, once more than two variates are correl ated, the choice of copulas often comes down to selection of the degrees-of-freedom parameter in the *t*-copula. In search for a wider selection of multivariate copulas we review a generalization of the *t*-copula and some copulas defined by Harry Joe. Generalizing the *t*-copula gives more flexibility in setting tail behavior. Possible applications include insurance losses by line, credit risk by issuer, and exchange rates. The Joe copulas are somewhat restricted in the range of correlations and tail dependencies that can be produced. However, both right- and left-positive tail dependence is possible, and the behavior is somewhat different from the *t*-copula.

KEYWORDS

Multivariate copulas, grouped t-copula, copula parameter estimation, copula fit testing, foreign exchange dependencies

1. Introduction

Copulas provide a convenient way to express multivariate distributions. Given the individual distribution functions $F_i(X_i)$ the multivariate distribution can be expressed as a copula function applied to the probabilities, i.e., $F(X_1,\ldots,X_n) = C[F_1(X_1),\ldots,F_n(X_n)]$. Venter [6] discusses many of the basic issues of using copulas with the heavy-tailed distributions of property and liability (P&L) insurance. One of the key concepts is that copulas can control where in the range of probabilities the dependence is strongest. Any copula C is itself a multivariate distribution function but one that applies only to uniform distributions on the unit square, unit cube,..., depending on dimension. The uniform $[0,1]$ variables are interpreted as probabilities from other distributions and are represented by U, V, etc.

One application is correlation in losses across lines of business. Lines tend to be weakly correlated in most cases but can be strongly correlated in extreme cases, like earthquakes. Belguise and Levi [1] study copulas applied to catastrophe losses across lines. With financial modeling growing in importance, other potential applications of copulas in insurance company management include the modeling of dependence between loss and loss expense, dependence among asset classes, dependence among currency exchange rates, and credit risk among reinsurers.

The upper and lower tail dependence coefficients of a copula provide quantification of tail strength. These can be defined using the right and left tail concentration functions R and L on $(0,1)$:

$$R(z) = \Pr(U > z \mid V > z) \qquad \text{and}$$

$$L(z) = \Pr(U < z \mid V < z).$$

The upper tail dependence coefficient is the limit of R as $z \to 1$, and the lower tail dependence coefficient is the limit of L as $z \to 0$. For the normal copula and many others, these coefficients are zero. This means that for extreme values the distributions are uncorrelated, so large-large or small-small combinations are not likely. However, this is somewhat misleading, as the slopes of the R and L functions for the normal and t-copulas can be very steep near the limits. Thus there can be a significant degree of dependence near the limits even when it is zero at the limit. Thus looking at $R(z)$ for z a bit less than 1 may be the best way to examine large loss dependencies.

While a variety of bivariate copulas is available, when more than two variables are involved the practical choice comes down to normal vs. t-copula. The normal copula is essentially the t-copula with high degrees of freedom (df), so the choice is basically what df to use in that copula. Venter [5] discusses the use of the t-copula in insurance. The t takes a correlation parameter for each pair of variates, and any correlation matrix can be used. The df parameter adds a common-shock effect. This can be seen in the simulation methodology, where first a vector of multivariate normal deviates is simulated then the vector is multiplied by a draw from a single inverse gamma distribution. The df parameter is the shape parameter of the latter distribution, which is more heavy-tailed with lower df. The factor drawn represents a common shock that hits all the normal variates with the same factor. This increases tail dependence in both the right and left tails but also increases the likelihood of anti-correlated results, so the overall correlation stays the same as in the normal copula.

Two problems for the t-copula are the symmetry between right and left tails and having only a single df parameter. The stronger tail dependence in insurance tends to happen in the right tail. Putting the same dependence in the left tail would be inaccurate, but it would probably have

little effect on risk measurement overall, which is largely affected by the right tail. The single df parameter is more of a restriction. It results in giving more tail dependence to more strongly correlated pairs. This is reasonable but is not always consistent with data.

Thus more flexible multivariate copulas would be useful. One alternative provided by Daul et al. [2] is the grouped t-copula, which uses different df parameters for different subgroups of variables, such as corporate bonds grouped by country. We introduce a special case of that called the individuated t-copula, or IT, which has a df parameter for each variable. Another direction is that of Joe [3], who develops a method for combining simple copulas to build up multivariate copulas. Three of these—called MM1, MM2, and MM3—have closed-form expressions. We investigate some of the properties of these copulas.

The t-copula for n variates has $(n^2 - n + 2)/2$ parameters. The MMC copulas have one more parameter for each variate and so have $(n^2 + n + 2)/2$ all together, while the IT has $(n^2 + n)/2$. This situation does not include enough parameters to have separate control of the strength of the tail dependency for every pair of variates, but it does add some flexibility. The IT copula generalizes the t so it can be tried any time the t is too limiting in the possibilities for tail behavior. The MMC copulas have a parameter for each pair of variates, but this does not give them full flexibility in matching a covariance matrix. When all the bivariate correlations are fairly low (below 50% in the trivariate case but decreasing with more dimensions) they can usually all be matched, but the higher they get the more similar they are forced to be. This would probably be fine for insurance losses by line of business, as they tend to have low correlations overall but can be related when the losses are large. In modeling them, some additional flexibility in tail behavior vs. overall correlation could be helpful, so this could be an application where the MMC copulas would have an advantage over the t-copula.

2. IT copula

This copula is most readily described by the simulation procedure from its parameters, which are a correlation matrix ρ and a parameter ν_n for each of the N variables. The simulation starts with the generation of a multivariate normal vector $\{z_n\}$ with correlation matrix ρ by the usual approach (Cholesky decomposition, etc.). Then a uniform $(0,1)$ variate u is drawn. The inverse chi-squared distribution quantile with probability u and df ν_n, denoted by $w_n = h_n(u)$, is calculated. Then $t_n = z_n[\nu_n/w_n]^{1/2}$ is t-distributed with ν_n df. To get the copula value, which is a probability, that t-distribution is applied to t_n. The only difference between this and simulation of the t-copula is the t uses the same inverse chi-square draw for each variate.

The chi-squared distribution is a special case of the gamma. The ratio w_n/ν_n is a scale transform of the chi-squared variate, so is a gamma variate. If the gamma density is parameterized to be proportional to $x^{\alpha-1}e^{-x/\beta}$, then w_n/ν_n has parameters $\beta = 2/\nu_n$ and $\alpha = \nu_n/2$. This is a distribution with mean 1. It can be simulated easily if an inverse gamma function is available, as in some spreadsheets. It is not necessary for the df to be an integer for this to work. If not an integer, a beta distribution can be used to calculate the t-distribution probability, as in Venter [5].

Although the simulation is straightforward, the copula density and probability functions are somewhat complicated. In terms of h_n, the inverse chi-squared function above, and denoting the matrix inverse of ρ as J and the inverse t-distribution of u_n with ν_n df as t_n, the copula

density at $(u_1,\ldots,u_n)$ can be shown to be:

$$c(\vec{u}) = \int_0^1 \frac{\prod_{n=1}^{N}\left\{\sqrt{h_n(y)}\Gamma(\nu_n/2)\left(1+\frac{t_n^2}{\nu_n}\right)^{(1+\nu_n)/2} \Big/ \Gamma\left(\frac{1+\nu_n}{2}\right)\right\}}{\sqrt{\det(\rho)(2\pi)^N}\exp\left(\sum_{n,m}\frac{J_{n,m}t_nt_m}{2}\sqrt{\frac{h_n(y)h_m(y)}{\nu_n\nu_m}}\right)}dy.$$

This has to be computed numerically unless the df parameters are all the same, in which case it reduces to the t-copula density.

Daul et al. [2] find that the correlations (Spearman's rho and Kendall's tau) are not exactly the same as for the t-copula, but are quite close. Also the bivariate t-copula right and left dependence coefficients are $S_{n+1}\{[(n+1)(1-\rho)/(1+\rho)]^{0.5}\}$ where S_{n+1} is the t-distribution survival function ($\Pr X > x$) with $n+1$ df. If we denote the bivariate standard normal survival function as $S(x,y) = \Pr(X > x, Y > y)$, then the tail dependence coefficients for X_m and X_n for the IT copula are:

$$\int_0^\infty S(c_n y^{1/\nu_n}, c_m y^{1/\nu_m})dy,$$

$$\text{where} \quad c_j = \sqrt{2}\left[\Gamma\left(\frac{1+\nu_j}{2}\right)\Big/\sqrt{4\pi}\right]^{1/\nu_j}.$$

These tend to be between the t-copula dependence coefficients for the two dfs but closer to that for the higher of the two dfs if these are very different.

Fitting the IT by maximum likelihood is possible but involves several numerical steps. Alternatively, the df parameter for the t-copula could be estimated for each pair of variables by matching tail behavior, as in Venter [5], and then individual dfs assigned to be consistent with this. Daul et al. [2] propose separate t-copulas for different groups of variables, which are combined into a single copula with the overall correlation matrix and the separate dfs.

3. MMC copulas

Joe's MM1, MM2, and MM3 copulas each have an overall strength parameter θ, a parameter for each pair of variables δ_{ij} (not the Kronecker delta), and add an additional parameter p_j, with $1/p_j \geq m-1$, for each of the m variables U_j, which gives the possibility of more control over the tails.

The parameters are then $\delta_{ij}$for $i<j, p_j$, for $j = 1,\ldots,m$, and θ. For each variable u_j, it is convenient to use the abbreviations $y_j = (-\ln u_j)^\theta$ and $w_j = p_j(u_j^{-\theta}-1)$. The δ_{ij} parameters would seem to allow for a correlation matrix, but it turns out that there are restrictions on how different the correlations can be, with the θ parameter exerting a lot of control. In effect, the parameters do not have as much freedom to fit to data as might be desired. The copula functions and dependency coefficients are given below. The density functions for the copulas needed for maximum likelihood estimation (MLE) are discussed in Appendix 1. Appendix 2 discusses numerical evaluation of MLE and rejection sampling (Tillé [4]) as a method for assessing standard errors.

MM1

Here $\delta_{ij} \geq 1$ and $\theta \geq 1$. The copula at the m-vector **u** is:

$$C(u) = \exp\left\{-\left[\sum_{j=1}^{m}(1-(m-1)p_j)y_j + \sum_{i<j}((p_iy_i)^{\delta_{ij}} + (p_jy_j)^{\delta_{ij}})^{1/\delta_{ij}}\right]^{1/\theta}\right\}$$

The bivariate i,j margin is:

$$C(u_i,u_j) = \exp\{-[(1-p_i)y_i + (1-p_j)y_j + ((p_iy_i)^{\delta_{ij}} + (p_jy_j)^{\delta_{ij}})^{1/\delta_{ij}}]^{1/\theta}\}$$

Lower tail dependence is zero. Upper tail dependence is given by:

$$\lambda_{ij} = 2 - [2 + (p_i^{\delta_{ij}} + p_j^{\delta_{ij}})^{1/\delta_{ij}} - p_i - p_j]^{1/\theta}$$

MM2

Here $\delta_{ij} > 0$ and $\theta > 0$. The copula at the m-vector $\mathbf{u}$ is:

$$C(u) = \left[\sum_{j=1}^{m} u_j^{-\theta} + 1 - m - \sum_{i<j}(w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}})^{-1/\delta_{ij}}\right]^{-1/\theta}$$

The i,j margin is:

$$C(u_i,u_j) = [u_i^{-\theta} + u_j^{-\theta} - 1 - (w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}})^{-1/\delta_{ij}}]^{-1/\theta}$$

Because both upper and lower tail dependence are positive for this copula, a subscript is used to distinguish them here. The upper tail dependence is:

$$\lambda_{ij,U} = (p_i^{-\delta_{ij}} + p_j^{-\delta_{ij}})^{-1/\delta_{ij}}$$

The lower tail dependence is:

$$\lambda_{ij,L} = [2 - (p_i^{-\delta_{ij}} + p_j^{-\delta_{ij}})^{-1/\delta_{ij}}]^{-1/\theta}$$

MM3

This starts with $\delta_{ij} > 0$ and $\theta > 1$ (Although Joe says $\theta > 0$ is allowed, we have had problems if $\theta < 1$.). The copula at $\mathbf{u}$ is

$$C(u) = \exp\left\{-\left[\sum_{j=1}^{m} y_j - \sum_{i<j}((p_iy_i)^{-\delta_{ij}} + (p_jy_j)^{-\delta_{ij}})^{-1/\delta_{ij}}\right]^{1/\theta}\right\}$$

The i,j margin is

$$C(u_i,u_j) = \exp\{-[y_i + y_j - ((p_iy_i)^{-\delta_{ij}} + (p_jy_j)^{-\delta_{ij}})^{-1/\delta_{ij}}]^{1/\theta}\}$$

The upper tail dependence is:

$$\lambda_{ij} = 2 - [2 - (p_i^{-\delta_{ij}} + p_j^{-\delta_{ij}})^{-1/\delta_{ij}}]^{1/\theta}$$

4. Range of possible correlations and dependencies

It turns out that not every possible correlation matrix can be matched by these copulas. The θ parameter determines a lot of what is possible for each copula. Tables 1–6 show the tail dependence and Spearman's ρ for selected parameters. For MM1 and MM3 the lower tail dependence is zero, so it is not shown. It is shown for MM2, but there the upper dependence does not depend on θ, so "any" is shown for θ.

5. Discussion

The tables in the previous section are bivariate relationships. However, since the p parameters can be no greater than $1/(m-1)$, the possible values of p, and so the range of possible correlations and tail dependencies, reduce as the dimension increases. The θ parameter i n general has a great deal of influence over what any of these copulas can do. As θ gets higher, the other parameters have very little influence. Thus when it is high, all pairs of variables will have similar correlations and tail dependencies. This is the case for any of the correlation coefficients (linear, rank, tau) so we use the term "correlation" generically. When θ is low, however, it is not possible to have very high correlations. Thus, in general, these copulas can only apply when all pairs of variables have similar tails and correlations.

Table 1. MM1 upper tail dependence

δ	θ:	1.037	1.037	1.037	1.037	1.111	1.111	1.111	1.111	1.333	1.333	1.333	1.333	2	2	2	2	4	4	4	4
	p_i	p_j:.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5
1.01	.005	.049	.049	.049	.049	.134	.134	.134	.134	.318	.318	.318	.318	.586	.586	.586	.586	.811	.811	.811	.811
1.01	.17	.049	.051	.052	.052	.134	.136	.137	.137	.318	.320	.320	.321	.586	.587	.587	.587	.811	.811	.811	.811
1.01	.335	.049	.052	.053	.054	.134	.137	.138	.139	.318	.320	.321	.322	.586	.587	.587	.588	.811	.811	.811	.812
1.01	.5	.049	.052	.054	.055	.134	.137	.139	.140	.318	.321	.322	.323	.586	.587	.588	.588	.811	.811	.812	.812
1.1	.005	.049	.051	.051	.051	.134	.135	.136	.136	.319	.319	.319	.320	.586	.586	.586	.587	.811	.811	.811	.811
1.1	.17	.051	.068	.076	.080	.135	.151	.158	.162	.319	.331	.336	.339	.586	.593	.596	.598	.811	.814	.815	.816
1.1	.335	.051	.076	.087	.095	.136	.158	.168	.176	.319	.336	.344	.349	.586	.596	.600	.603	.811	.815	.817	.818
1.1	.5	.051	.080	.095	.106	.136	.162	.176	.185	.320	.339	.349	.357	.587	.598	.603	.608	.811	.816	.818	.820
2	.005	.052	.054	.054	.054	.136	.138	.138	.138	.320	.321	.321	.321	.587	.588	.588	.588	.811	.812	.812	.812
2	.17	.054	.143	.171	.183	.138	.218	.243	.254	.321	.381	.400	.409	.588	.621	.632	.637	.812	.826	.831	.832
2	.335	.054	.171	.234	.269	.138	.243	.300	.331	.321	.400	.444	.468	.588	.632	.657	.671	.812	.831	.841	.847
2	.5	.054	.183	.269	.325	.138	.254	.331	.382	.321	.409	.468	.507	.588	.637	.671	.693	.812	.832	.847	.857
11	.005	.053	.054	.054	.054	.138	.138	.138	.138	.321	.321	.321	.321	.587	.588	.588	.588	.811	.812	.812	.812
11	.17	.054	.199	.209	.209	.138	.268	.277	.277	.321	.419	.427	.427	.588	.643	.647	.647	.812	.835	.837	.837
11	.335	.054	.209	.344	.365	.138	.277	.399	.417	.321	.427	.520	.534	.588	.647	.701	.709	.812	.837	.860	.864
11	.5	.054	.209	.365	.491	.138	.277	.417	.532	.321	.427	.534	.623	.588	.647	.709	.762	.812	.837	.864	.887

Table 2. MM1 Spearman's ρ

δ	θ:	1.037	1.037	1.037	1.037	1.111	1.111	1.111	1.111	1.333	1.333	1.333	1.333	2	2	2	2	4	4	4	4
	p_i	p_j:.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5
1.01	.01	.053	.054	.054	.054	.149	.149	.149	.149	.364	.364	.364	.364	.682	.682	.682	.682	.913	.913	.913	.913
1.01	.17	.054	.056	.057	.057	.149	.151	.152	.152	.364	.365	.366	.366	.682	.683	.684	.684	.913	.913	.913	.913
1.01	.335	.054	.057	.058	.059	.149	.152	.153	.154	.364	.366	.367	.368	.682	.684	.684	.685	.913	.913	.913	.913
1.01	.5	.054	.057	.059	.061	.149	.152	.154	.155	.364	.366	.368	.369	.682	.684	.685	.685	.913	.913	.913	.913
1.1	.01	.054	.056	.056	.056	.149	.151	.151	.151	.364	.365	.365	.366	.682	.683	.683	.683	.913	.913	.913	.913
1.1	.17	.056	.075	.083	.088	.151	.168	.176	.181	.365	.379	.385	.389	.683	.691	.694	.696	.913	.915	.916	.917
1.1	.335	.056	.083	.096	.105	.151	.176	.187	.196	.365	.385	.394	.401	.683	.694	.699	.702	.913	.916	.917	.918
1.1	.5	.056	.088	.105	.117	.151	.181	.196	.207	.366	.389	.401	.409	.683	.696	.702	.707	.913	.917	.918	.920
2	.01	.056	.060	.060	.060	.151	.155	.155	.155	.366	.369	.369	.369	.683	.685	.685	.686	.913	.913	.914	.914
2	.17	.060	.148	.182	.203	.155	.235	.267	.286	.369	.430	.455	.470	.685	.717	.730	.739	.913	.923	.926	.929
2	.335	.060	.182	.245	.287	.155	.267	.324	.362	.369	.455	.498	.528	.685	.730	.753	.768	.914	.926	.933	.937
2	.5	.060	.203	.287	.348	.155	.286	.362	.417	.369	.470	.528	.570	.686	.739	.768	.790	.914	.929	.937	.943
11	.01	.057	.060	.061	.061	.152	.155	.155	.155	.366	.369	.369	.369	.684	.686	.686	.686	.913	.914	.914	.914
11	.17	.060	.179	.223	.246	.155	.263	.303	.325	.369	.451	.482	.500	.686	.727	.744	.754	.914	.925	.930	.933
11	.335	.061	.223	.312	.369	.155	.303	.383	.435	.369	.482	.542	.582	.686	.744	.774	.794	.914	.930	.938	.944
11	.5	.061	.246	.369	.457	.155	.325	.435	.514	.369	.500	.582	.641	.686	.754	.794	.823	.914	.933	.944	.952

Table 3. MM2 upper tail dependence and MM2 lower tail dependence

		MM2 Upper Tail Dependence				MM2 Lower Tail Dependence															
δ	θ:	any	any	any	any	.111	.111	.111	.111	.333	.333	.333	.333	1	1	1	1	3	3	3	3
	p_i	p_j:.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5
.25	.005	.000	.001	.002	.002	.002	.002	.002	.002	.125	.125	.125	.125	.500	.500	.500	.500	.794	.794	.794	.794
.25	.17	.001	.011	.015	.018	.002	.002	.002	.002	.125	.127	.128	.128	.500	.503	.504	.504	.794	.795	.796	.796
.25	.335	.002	.015	.021	.025	.002	.002	.002	.002	.125	.128	.129	.130	.500	.504	.505	.506	.794	.796	.796	.797
.25	.5	.002	.018	.025	.031	.002	.002	.002	.002	.125	.128	.130	.131	.500	.504	.506	.508	.794	.796	.797	.798
1	.005	.003	.005	.005	.005	.002	.002	.002	.002	.125	.126	.126	.126	.501	.501	.501	.501	.794	.794	.794	.794
1	.17	.005	.085	.113	.127	.002	.003	.003	.004	.126	.142	.149	.152	.501	.522	.530	.534	.794	.805	.809	.811
1	.335	.005	.113	.168	.201	.002	.003	.004	.005	.126	.149	.163	.172	.501	.530	.546	.556	.794	.809	.817	.822
1	.5	.005	.127	.201	.250	.002	.004	.005	.006	.126	.152	.172	.187	.501	.534	.556	.571	.794	.811	.822	.830
4	.005	.004	.005	.005	.005	.002	.002	.002	.002	.126	.126	.126	.126	.501	.501	.501	.501	.794	.794	.794	.794
4	.17	.005	.143	.167	.169	.002	.004	.004	.004	.126	.156	.162	.163	.501	.538	.546	.546	.794	.814	.817	.817
4	.335	.005	.167	.282	.320	.002	.004	.008	.009	.126	.162	.197	.211	.501	.546	.582	.595	.794	.817	.835	.841
4	.5	.005	.169	.320	.420	.002	.004	.009	.016	.126	.163	.211	.254	.501	.546	.595	.633	.794	.817	.841	.859
16	.005	.005	.005	.005	.005	.002	.002	.002	.002	.126	.126	.126	.126	.501	.501	.501	.501	.794	.794	.794	.794
16	.17	.005	.163	.170	.170	.002	.004	.004	.004	.126	.161	.163	.163	.501	.544	.546	.546	.794	.816	.818	.818
16	.335	.005	.170	.321	.335	.002	.004	.009	.010	.126	.163	.211	.217	.501	.546	.596	.601	.794	.818	.841	.844
16	.5	.005	.170	.335	.479	.002	.004	.010	.023	.126	.163	.217	.284	.501	.546	.601	.657	.794	.818	.844	.870

Table 4. MM2 Spearman's ρ

δ	θ:	0.037	0.037	0.037	0.037	0.111	0.111	0.111	0.111	0.333	0.333	0.333	0.333	1	1	1	1	3	3	3	3
	p_i	p_j:.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5
.25	.005	.028	.029	.029	.029	.079	.080	.081	.081	.213	.214	.214	.214	.479	.479	.479	.480	.787	.787	.787	.787
.25	.17	.029	.039	.043	.047	.080	.090	.094	.097	.214	.222	.226	.229	.479	.485	.488	.490	.787	.790	.791	.792
.25	.335	.029	.043	.050	.055	.081	.094	.101	.106	.214	.226	.232	.236	.479	.488	.492	.495	.787	.791	.793	.794
.25	.5	.029	.047	.055	.062	.081	.097	.106	.112	.214	.229	.236	.241	.480	.490	.495	.499	.787	.792	.794	.796
1	.005	.030	.034	.034	.034	.081	.085	.085	.085	.214	.218	.218	.218	.480	.482	.483	.483	.787	.788	.788	.789
1	.17	.034	.113	.144	.164	.085	.160	.190	.209	.218	.284	.310	.326	.482	.527	.546	.557	.788	.808	.815	.821
1	.335	.034	.144	.200	.238	.085	.190	.243	.280	.218	.310	.356	.387	.483	.546	.577	.599	.788	.815	.829	.838
1	.5	.034	.164	.238	.291	.085	.209	.280	.331	.218	.326	.387	.432	.483	.557	.599	.629	.789	.821	.838	.851
4	.005	.031	.034	.035	.035	.082	.086	.086	.086	.215	.218	.219	.219	.480	.483	.483	.483	.787	.789	.789	.789
4	.17	.034	.152	.195	.219	.086	.197	.239	.262	.218	.315	.352	.372	.483	.548	.574	.588	.789	.816	.827	.833
4	.335	.035	.195	.282	.338	.086	.239	.322	.375	.219	.352	.423	.469	.483	.574	.621	.652	.789	.827	.846	.859
4	.5	.035	.219	.338	.424	.086	.262	.375	.456	.219	.372	.469	.539	.483	.588	.652	.698	.789	.833	.859	.878
16	.005	.031	.034	.035	.035	.082	.086	.086	.086	.215	.219	.219	.219	.480	.483	.483	.483	.787	.789	.789	.789
16	.17	.034	.157	.202	.225	.086	.202	.245	.268	.219	.320	.357	.377	.483	.551	.577	.592	.789	.817	.828	.835
16	.335	.035	.202	.294	.352	.086	.245	.333	.388	.219	.357	.432	.480	.483	.577	.627	.659	.789	.828	.848	.862
16	.5	.035	.225	.352	.443	.086	.268	.388	.474	.219	.377	.480	.554	.483	.592	.659	.708	.789	.835	.862	.882

Table 5. MM3 upper tail dependence

δ	θ:	1.037	1.037	1.037	1.037	1.111	1.111	1.111	1.111	1.333	1.333	1.333	1.333	2	2	2	2	4	4	4	4
	p_i	p_j:.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5
.25	.005	.049	.050	.050	.050	.134	.135	.135	.135	.318	.319	.319	.319	.586	.586	.586	.586	.811	.811	.811	.811
.25	.17	.050	.059	.063	.065	.135	.143	.146	.149	.319	.325	.327	.329	.586	.590	.591	.592	.811	.812	.813	.813
.25	.335	.050	.063	.069	.073	.135	.146	.152	.155	.319	.327	.331	.334	.586	.591	.593	.595	.811	.813	.814	.815
.25	.5	.050	.065	.073	.078	.135	.149	.155	.160	.319	.329	.334	.338	.586	.592	.595	.597	.811	.813	.815	.815
1	.005	.051	.053	.054	.054	.136	.138	.138	.138	.320	.321	.321	.321	.587	.588	.588	.588	.811	.812	.812	.812
1	.17	.053	.129	.155	.168	.138	.205	.229	.241	.321	.372	.390	.399	.588	.616	.626	.631	.812	.824	.828	.830
1	.335	.054	.155	.207	.238	.138	.229	.275	.303	.321	.390	.425	.446	.588	.626	.646	.659	.812	.828	.837	.842
1	.5	.054	.168	.238	.285	.138	.241	.303	.345	.321	.399	.446	.478	.588	.631	.659	.677	.812	.830	.842	.850
4	.005	.053	.054	.054	.054	.137	.138	.138	.138	.321	.321	.321	.321	.587	.588	.588	.588	.811	.812	.812	.812
4	.17	.054	.184	.207	.209	.138	.254	.275	.277	.321	.409	.425	.426	.588	.637	.646	.647	.812	.833	.836	.837
4	.335	.054	.207	.315	.351	.138	.275	.372	.405	.321	.425	.499	.524	.588	.646	.689	.704	.812	.836	.855	.862
4	.5	.054	.209	.351	.446	.138	.277	.405	.491	.321	.426	.524	.591	.588	.647	.704	.743	.812	.837	.862	.879
16	.005	.053	.054	.054	.054	.138	.138	.138	.138	.321	.321	.321	.321	.587	.588	.588	.588	.812	.812	.812	.812
16	.17	.054	.202	.209	.209	.138	.271	.277	.277	.321	.422	.427	.427	.588	.645	.647	.647	.812	.836	.837	.837
16	.335	.054	.209	.352	.365	.138	.277	.406	.418	.321	.427	.525	.534	.588	.647	.704	.710	.812	.837	.862	.864
16	.5	.054	.209	.365	.501	.138	.277	.418	.541	.321	.427	.534	.630	.588	.647	.710	.767	.812	.837	.864	.889

Table 6. MM3 Spearman's ρ

δ	θ:	1.037	1.037	1.037	1.037	1.111	1.111	1.111	1.111	1.333	1.333	1.333	1.333	2	2	2	2	4	4	4	4
	p_i	p_j:.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5	.005	.17	.335	.5
.25	.005	.054	.055	.055	.055	.148	.150	.150	.150	.363	.365	.365	.365	.682	.683	.683	.683	.912	.913	.913	.913
.25	.17	.055	.065	.069	.072	.150	.159	.163	.166	.365	.372	.375	.377	.683	.687	.689	.690	.913	.914	.914	.915
.25	.335	.055	.069	.076	.081	.150	.163	.169	.174	.365	.375	.380	.384	.683	.689	.691	.693	.913	.914	.915	.916
.25	.5	.055	.072	.081	.087	.150	.166	.174	.180	.365	.377	.384	.388	.683	.690	.693	.696	.913	.915	.916	.917
1	.005	.051	.060	.060	.060	.146	.154	.155	.155	.362	.368	.369	.369	.681	.685	.685	.685	.912	.913	.914	.914
1	.17	.060	.137	.168	.187	.154	.225	.253	.271	.368	.423	.445	.459	.685	.713	.725	.733	.913	.922	.925	.927
1	.335	.060	.168	.222	.259	.155	.253	.303	.336	.369	.445	.483	.509	.685	.725	.745	.759	.914	.925	.931	.934
1	.5	.060	.187	.259	.311	.155	.271	.336	.384	.369	.459	.509	.545	.685	.733	.759	.778	.914	.927	.934	.940
4	.005	.050	.060	.061	.061	.146	.155	.155	.155	.366	.369	.369	.369	.683	.686	.686	.686	.913	.914	.914	.914
4	.17	.060	.175	.217	.241	.155	.259	.298	.320	.369	.448	.479	.496	.686	.726	.742	.752	.914	.925	.929	.932
4	.335	.061	.217	.302	.357	.155	.298	.374	.424	.369	.479	.536	.574	.686	.742	.771	.790	.914	.929	.937	.943
4	.5	.061	.241	.357	.440	.155	.320	.424	.499	.369	.496	.574	.630	.686	.752	.790	.818	.914	.932	.943	.951
16	.005	.050	.060	.061	.061	.145	.155	.155	.155	.366	.369	.369	.369	.684	.686	.686	.686	.913	.914	.914	.914
16	.17	.060	.180	.224	.247	.155	.264	.304	.326	.369	.451	.483	.501	.686	.727	.744	.754	.914	.925	.930	.933
16	.335	.061	.224	.313	.370	.155	.304	.384	.436	.369	.483	.543	.583	.686	.744	.774	.794	.914	.930	.938	.944
16	.5	.061	.247	.370	.459	.155	.326	.436	.516	.369	.501	.583	.642	.686	.754	.794	.824	.914	.933	.944	.952

Figure 1. MM1 R as a function of rho by theta, $m = 3$

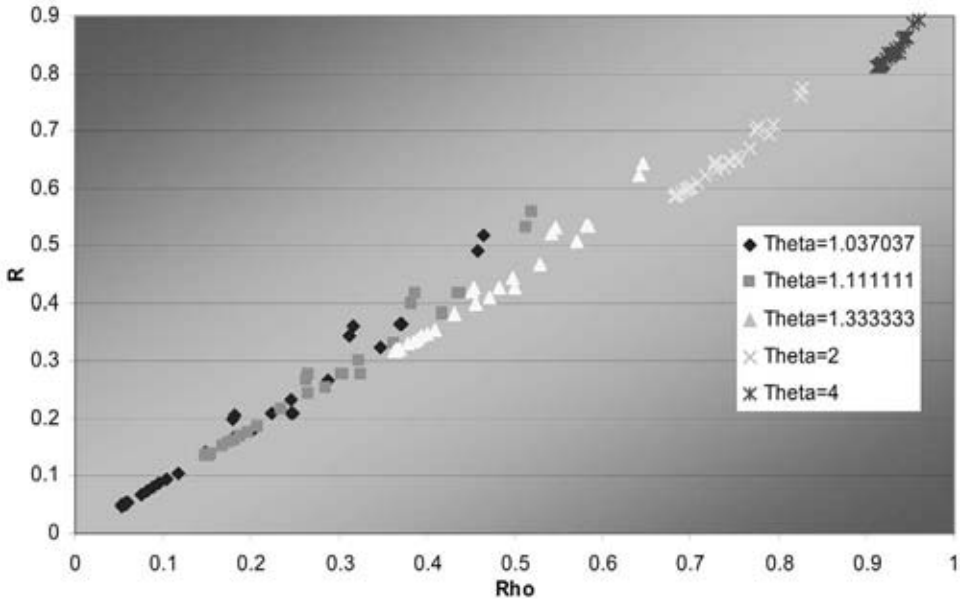
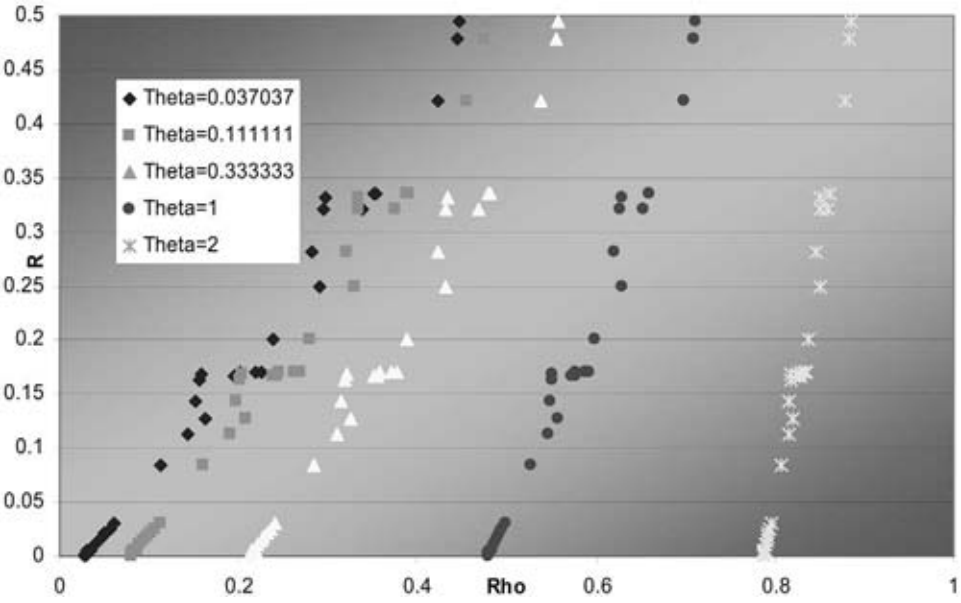


Figure 2. MM2 R as a function of rho by theta, $m = 3$



Also the right tail dependence R and the correlation are closely related. Scatter plots of R as a function of ρ for different values of θ are shown for the cases $m = 3$ and $m = 7$ for the MM1 and MM2 copulas and $m = 7$ for MM3 in Figures 1–5. These points are taken from the above tables so just give a suggestion of the spread in R available for the same values of ρ and θ. Even so, it is clear that some degree of variability of R is possible for pairs within a single copula even for the same rank correlation. This is in contrast to the t-copula for which the correlation and df determine the tail dependence uniquely.

Figure 3. MM1 R as a function of rho by theta, $m = 7$

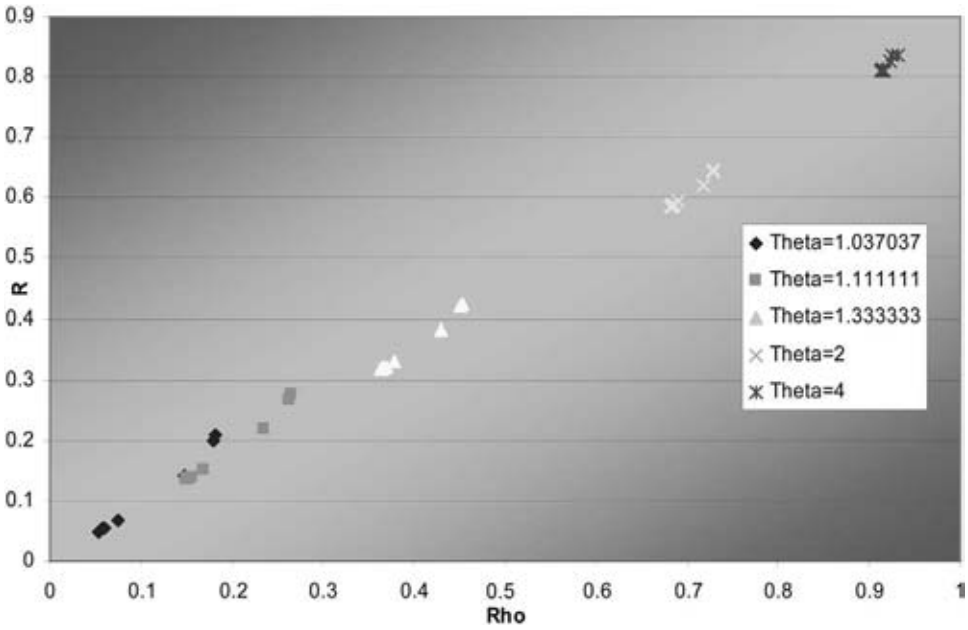
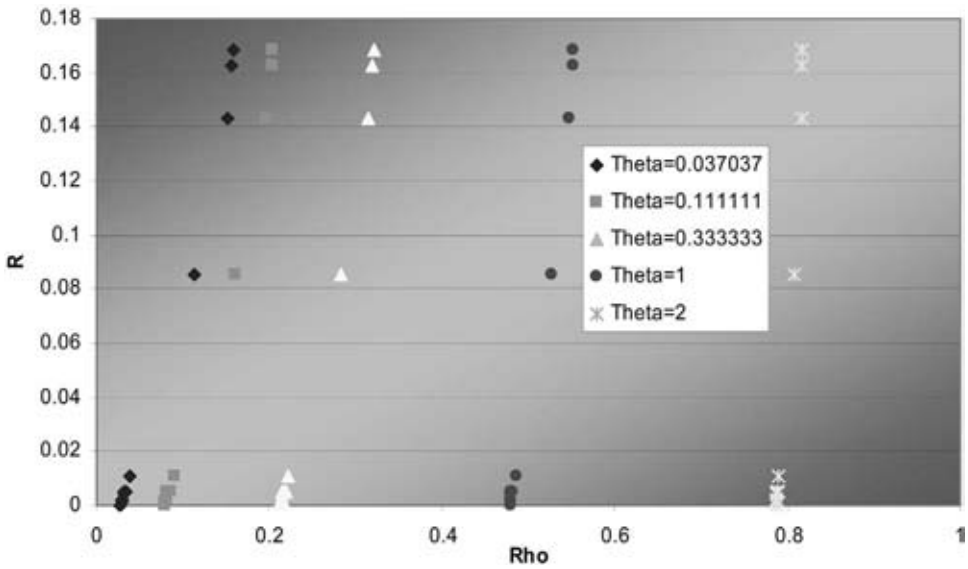


Figure 4. MM2 R as a function of rho by theta, $m = 7$



For MM2, R is not a function of θ, so a wider range of Rs is possible for any θ. The possible range of ρ for each θ is about the same for each copula. This range decreases with higher ms, however, due to the restricted range of p.

The relationship for MM3 in Figure 5 is similar to that for MM1 in Figure 1, but MM3 might have a bit more possible spread in R.

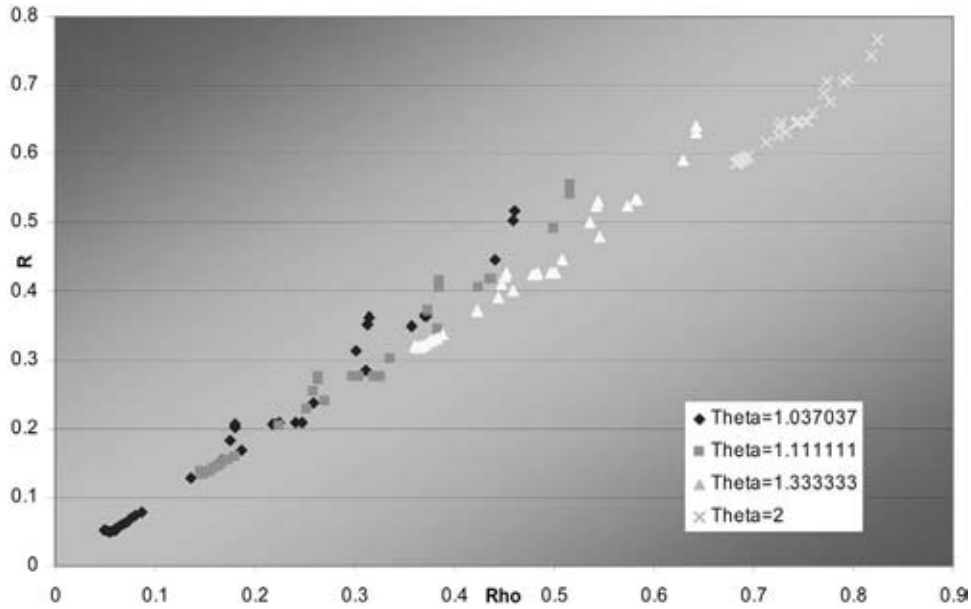
6. Fitting MM1 and MM2

As an example, we tried fitting the MM1 and MM2 copulas to currency rate changes. Currency data was chosen because it is publicly available, not because it is fit better by the MMC copulas. But by fitting to actual data, some of the shape differences among the copulas can be

Table 7. Parameters for the various fitting methods

	MM2		MM1			t	
	SSE	MLE bi	SSE	MLE bi	MLE tri	MLE bi	MLE tri
δ_{12}	2.62588	1.50608	3.69513	2.14275	2.1094	0.491	0.490
δ_{13}	0.80055	0.43963	1.52005	1.19832	1.1152	0.262	0.266
δ_{23}	3.2E-07	0.0103	1	1	1	0.097	0.097
p_1	0.49881	0.37649	0.49963	0.40533	0.37175		
p_2	0.5	0.5	0.5	0.5	0.5		
p_3	0.26236	0.49625	0.29097	0.5	0.5		
θ	0.19599	0.2209	1.08308	1.0939	1.1234	20.53	20.95

Figure 5. MM2 R as a function of rho by theta, $m = 3$



illustrated, as can fitting methods. The data consists of monthly changes in the US $ exchange rate for the Swedish, Japanese and Canadian currencies from 1971 through September 2005. The data come from the Fred database of the St. Louis Federal Reserve. The Fred exchange rates were converted to monthly change factors, with a factor greater than 1.0 representing an appreciation of the US $ compared to that currency. These factors are comparative movements against the dollar, so if an event strengthens or weakens the dollar in particular, all the currencies would be expected to move in the same direction. Thus there could be common shocks.

The MMC copula densities get increasingly difficult to calculate as the dimension increases. For this reason, some alternatives to MLE were explored. One alternative is to maximize the product of the bivariate likelihood functions, which just requires the bivariate densities. Since the copulas are defined by the copula functions, an even easier fit is to minimize the distance between the empirical and fitted copulas, measured by sum of squared errors (SSE). Numerical differentiation, as discussed in Appendix 2, is another way to calculate the multivariate MLE but was not tried here.

The parameters for the various fitting methods are shown in Table 7. The product of the bivariate likelihood estimates for MM1 are quite close to the full trivariate MLE. It should be noted, however, that there are many local max-

Figure 6. Sweden—Japan J function

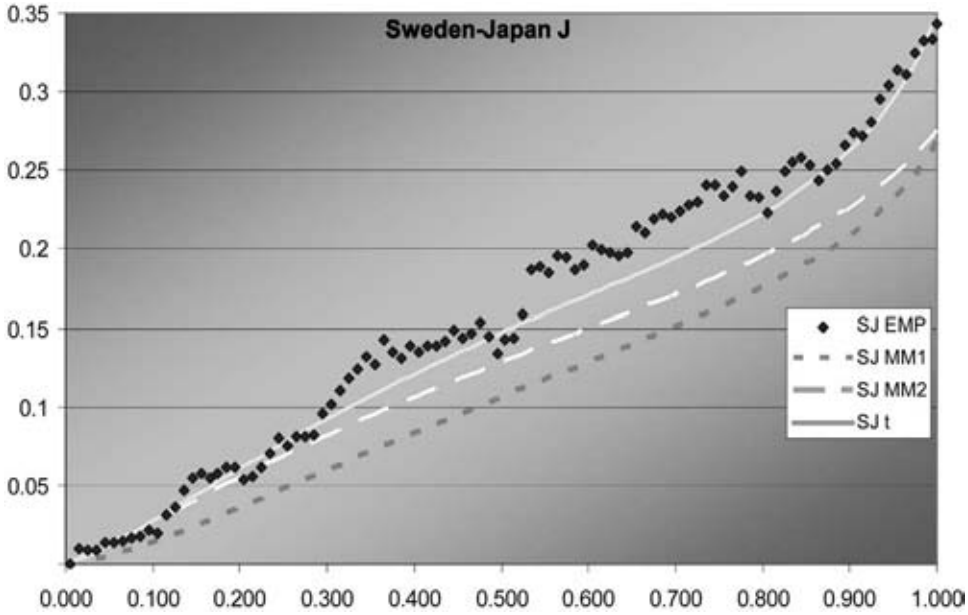
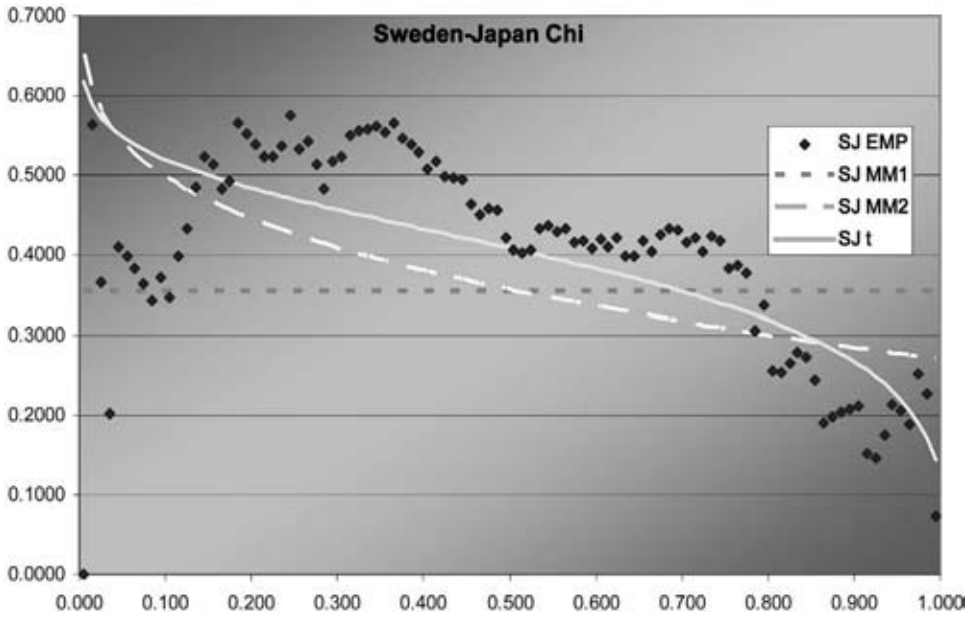


Figure 7. Sweden—Japan χ function



ima for both likelihood functions, so we cannot be absolutely sure that these are the global maxima. Only the bivariate estimates were done for MM2. The SSE parameters are quite a bit different from the MLEs. Thus, the product of the bivariate MLEs appears to be the more promising short-cut method.

The t parameters were estimated by maximizing the trivariate and product of bivariate likelihood functions for comparison. We use a beta distribution version of the t that allows fractional degrees of freedom. Comparisons of fits were done using graphs of the empirical and fitted J and χ functions in Figures 6–11. These functions

Figure 8. Sweden—Canada J function

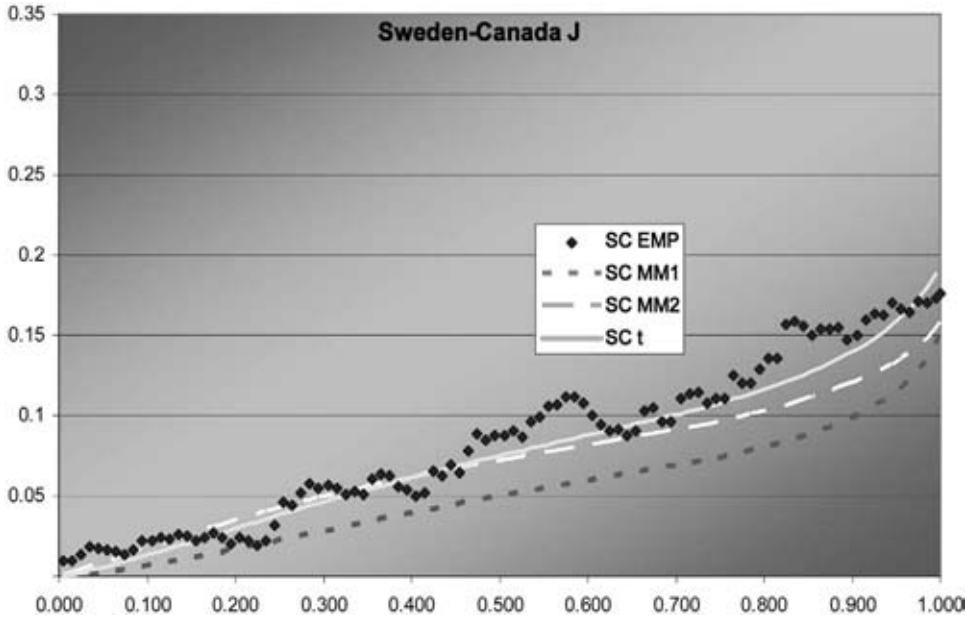
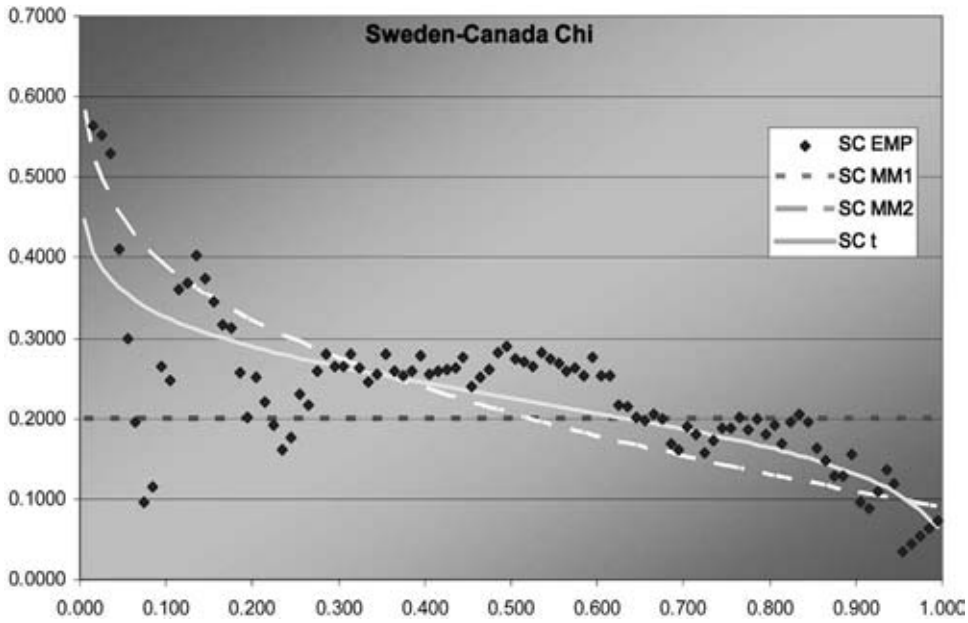


Figure 9. Sweden—Canada χ function



are defined as:

$$J(z) = -z^2 + 4\int_0^z \int_0^z C(u,v)c(u,v)dv\,du/C(z,z) \quad \text{and}$$

$$\chi(z) = 2 - \ln[C(z,z)]/\ln z.$$

As $z \to 1$ these approach Kendall's τ and the upper tail dependence R, respectively. They also can be compared to see the fit in the left tail, which is important for currency movements.

In two of the three J graphs, the t-copula is clearly the best fit, but MM2 is close for Sweden-Canada. In the third, MM2 is the best. MM1 is always worse for this data. The t is also best in two of the three χ graphs, but not much better than MM2 for Sweden-Canada. In the third

Figure 10. Japan—Canada J function

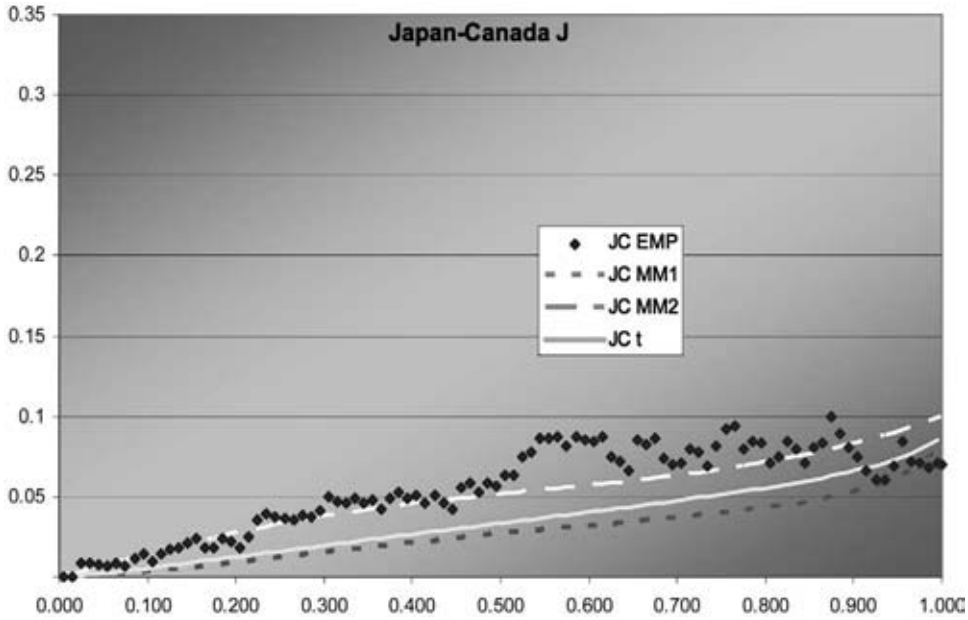
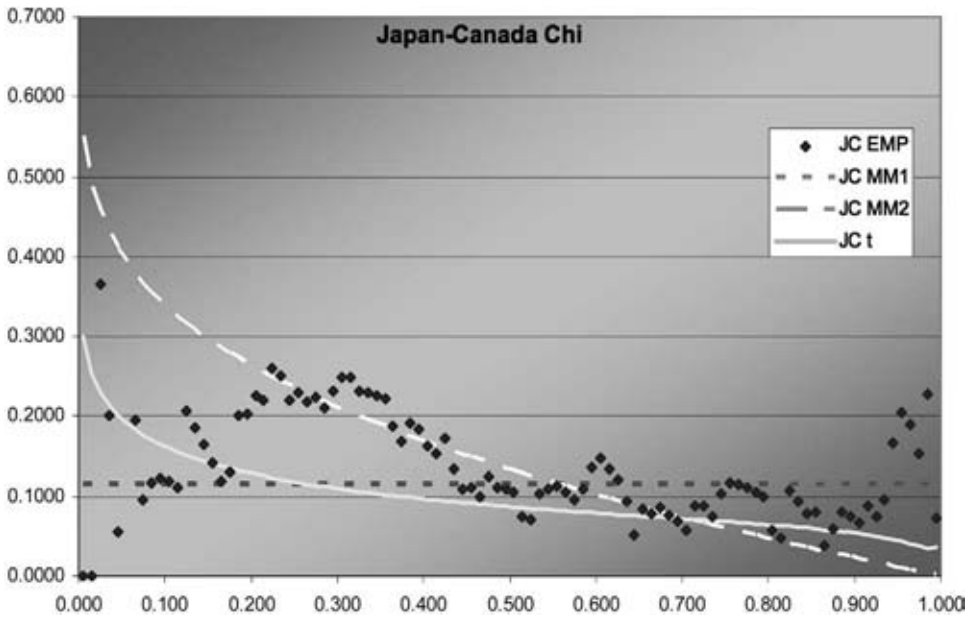


Figure 11. Japan—Canada χ function



graph, each of the three is best in some range but MM1 is probably best. Overall, MM2 seems to provide almost as good a fit as t, but not quite.

Although this data set is fit best by the t-copula, that does not negate the value of having more than one multivariate copula available. The example explores methods of fitting multivariate copulas and comparing their fits, and illustrates the different shapes of the descriptive functions that can be produced by the different copulas. For exchange rates in particular, it may be worth exploring fitting the logs of the change factors by multivariate normal and t-distributions, as re-

ciprocals of the factors are of interest and could have the same distribution.

7. Summary

The IT copula has potential for better fits when the t-copula tails differ. Trying to fit this to some actual data seems worthwhile.

The closed-form MMC copulas do not give full flexibility in the range of the bivariate correlations that can occur in a single multivariate copula. These copulas would be most appropriate when the empirical correlations are all fairly small or fairly close to each other. This gets more so as the number of dimensions increases. These copulas all have somewhat different shapes, and differ from the t-copula as well, so they could be useful with the right data. An interesting application may be to insurance losses by line of business, as their correlations are all relatively low and the tail dependences may not have the t-copula property of strictly growing with correlation. The correlations of losses across lines may not have the t's symmetric right and left tails either, but this is less of a concern, as putting too much correlation into small losses is not likely to critically affect the properties of the distribution of the sum of the lines.

Joe also defines MM4 and MM5 copulas. These do not have closed-form expressions, but he asserts that they have a wide range of possible dependence. To build a larger repertoire of multivariate copulas, it may be worth developing algorithms for calculation and fitting these and trying them on live data.

Appendix 1. MMC density functions

For maximum likelihood estimation, it would be desirable to have expressions for the copula densities, which are mixed partial derivatives of the copula C functions. Although it is difficult to write down general formulas for the densities, some broad outlines can be developed. Recall the product formula for derivatives $(ab)' = a'b + b'a$. Then $(abc)' = (ab)'c + abc' = a'bc + ab'c + abc'$. Similarly $(abcd)' = a'bcd + ab'cd + abc'd + abcd'$, etc. For any function F of the vector $\mathbf{u}$, consider the simple multivariate function $B(\mathbf{u}) = F(\mathbf{u})^a$. Denoting partial derivatives with respect to the elements of $\mathbf{u}$ by subscripts, we have successively:

$$B = F^a.$$

$$B_i = aF^{a-1}F_i.$$

$$B_{ij} = a(a-1)F^{a-2}F_iF_j + aF^{a-1}F_{ij}.$$

$$B_{ijk} = a(a-1)(a-2)F^{a-3}F_iF_jF_k + a(a-1)F^{a-2} \times (F_iF_{jk} + F_jF_{ik} + F_kF_{ij}) + aF^{a-1}F_{ijk}.$$

$$\begin{aligned} B_{ijkl} = {} & a(a-1)(a-2)(a-3)F^{a-4}F_iF_jF_kF_l \\ & + a(a-1)(a-2)F^{a-3}(F_{ij}F_kF_l + F_{ik}F_jF_l \\ & \quad + F_{il}F_jF_k + F_{jk}F_iF_l + F_{jl}F_iF_k + F_{kl}F_iF_j) \\ & + a(a-1)F^{a-2}(F_iF_{jkl} + F_jF_{ikl} + F_kF_{ijl} \\ & \quad + F_lF_{ijk} + F_{ij}F_{kl} + F_{ik}F_{jl} + F_{il}F_{jk}) \\ & + aF^{a-1}F_{ijkl}. \end{aligned}$$

This is the form of the MM2 C function. Although a pattern is emerging in the subscripts, it seems difficult to write down a general rule for an arbitrary mixed partial.

A similar exercise can be carried out for $C(\mathbf{u}) = e^{-B(\mathbf{u})}$.

$$C = e^{-B}.$$

$$C_1 = -CB_1.$$

$$C_{12} = C(B_1B_2 - B_{12}).$$

$$C_{123} = C(B_1B_{23} + B_2B_{13} + B_3B_{12} - B_1B_2B_3 - B_{123}).$$

$$\begin{aligned} C_{1234} = C(&B_1B_{234} + B_2B_{134} + B_3B_{124} + B_4B_{123} \\ &+ B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23} \\ &- B_{12}B_3B_4 - B_{13}B_2B_4 - B_{14}B_2B_3 \\ &- B_{23}B_1B_4 - B_{24}B_1B_3 - B_{34}B_1B_2 \\ &+ B_1B_2B_3B_4 - B_{1234}). \end{aligned}$$

This is the form of the MM1 and MM3 C with F^a replacing B.

To calculate the derivatives let $x_j = y_j' = \theta(-\ln u_j)^{\theta-1}/u_j$, and $t_j = w_j' = \theta p_j u_j^{-\theta-1}$. The case $m = 3$ is not too bad. First for MM1:

$$F(u) = \sum_{j=1}^{3}(1-2p_j)y_j + \sum_{i<j}((p_iy_i)^{\delta_{ij}} + (p_jy_j)^{\delta_{ij}})^{1/\delta_{ij}}$$

Taking the first derivative:

$$\begin{aligned} F_i(u) = &(1-2p_j)y_jx_i + (p_iy_i)^{\delta_{ij}-1}p_ix_i \\ &\times \sum_{j\neq i}((p_iy_i)^{\delta_{ij}} + (p_jy_j)^{\delta_{ij}})^{1/\delta_{ij}-1} \end{aligned}$$

This would be almost the same for the bivariate margin. Then:

$$\begin{aligned} F_{ij}(u) = &(1-\delta_{ij})(p_ip_jy_iy_j)^{\delta_{ij}-1}p_ip_jx_ix_j \\ &\times ((p_iy_i)^{\delta_{ij}} + (p_jy_j)^{\delta_{ij}})^{1/\delta_{ij}-2} \end{aligned}$$

This is the same for the bivariate margin. From this it is clear that $F_{123} = 0$. Then taking $a = 1/\theta$ and plugging in these values of F_i and F_{ij} will give all values of B_i, B_{ij}, and B_{123}, which can then be plugged in the formula for C_{123} to give the density. From the formulas for C_{12} the bivariate density is similar.

MM2 is even easier, as the C formulas are not needed. What is needed is:

$$F(u) = \sum_{j=1}^{3}u_j^{-\theta} - 2 - \sum_{i<j}(w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}})^{-1/\delta_{ij}}$$

This gives:

$$F_i(u) = -\theta u_i^{-\theta-1} - w_i^{-\delta_{ij}-1}t_i\sum_{j\neq i}(w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}})^{-1/\delta_{ij}-1}$$

And:

$$F_{ij}(u) = -(1+\delta_{ij})(w_iw_j)^{-\delta_{ij}-1}t_it_j(w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}})^{-1/\delta_{ij}-2}$$

Again $F_{123} = 0$. Then setting $a = -1/\theta$ gives B_{123}, which is the density.

This gets harder and harder as the dimension of $\mathbf{u}$ increases. Then numerical differentiation becomes increasingly attractive.

Appendix 2. Numerical densities

Background and notation

The discussion is oriented to copulas, i.e., smooth, parametric, cumulative distribution functions on the unit hypercube.

Let $X \in \Omega = [0,1]^d$ with cdf $F(X \mid \theta)$. Sample points are denoted x alone or $x^{(i)}$ if a sequence needs to be indexed. Components of x are x_j.

From CDF to measure

Define the hypercube $H(x,\delta)$ as

$$\begin{aligned} &([x_1 - \delta/2, x_1 + \delta/2] \times [x_2 - \delta/2, x_2 + \delta/2] \times \cdots \\ &\quad \times [x_d - \delta/2, x_d + \delta/2]) \cap \Omega \end{aligned}$$

Without loss of generality, we may assume a generic H is wholly contained in Ω. We can identify the set of vertices of $H(x,\delta)$ with the set $\mathbf{S} = \{-1,1\}^d$ by the mapping

$$\begin{aligned} s &= \langle s_1, s_2, \ldots, s_d\rangle \\ &\mapsto \langle x_1 + s_1\cdot\delta/2, x_2 + s_2\cdot\delta/2, \ldots, x_d + s_d\cdot\delta/2\rangle \\ &= C(x,\delta,s). \end{aligned}$$

Let $g(s) = \prod_j s_j$ be the sign of s.

Define the probability measure $\mu(H \mid \theta)$ as $\Pr\{X \in H \mid \theta\}$.

THEOREM

$$\mu(H(x,\delta) \mid \theta) = \sum_{s\in S} g(s)\cdot F(C(x,\delta,s) \mid \Theta).$$

The probability of X being in a hypercube is the sum of the distribution function at positive corners minus its value at the negative corners. For $d = 1$ this is just the difference between the function at

the top and bottom of the interval. For $d = 2$, this is the sum of the function at the upper right and lower left corners minus the function at each of the other corners. The latter two functions cover an overlapping area that gets subtracted twice, which is why you have to add back the function at the lower left. In general the proof requires keeping track of what are the positive and negative corners.

PROOF For $0 \le N \le d$, define the hypersolids

$$\begin{aligned} H_N &= ([x_1 - \delta/2, x_1 + \delta/2] \times \cdots \\ &\quad \times [x_N - \delta/2, x_N + \delta/2] \times [0, x_{N+1} + \delta/2] \\ &\quad \times [0, x_{N+2} + \delta/2] \times \cdots \times [0, x_d + \delta/2]) \\ H_N^- &= ([x_1 - \delta/2, x_1 + \delta/2] \times \cdots \\ &\quad \times [x_N - \delta/2, x_N + \delta/2] \times [0, x_{N+1} - \delta/2] \\ &\quad \times [0, x_{N+2} + \delta/2] \times \cdots \times [0, x_d + \delta/2]) \end{aligned}$$

and associated vertex index sets

$$S_N = \{\langle \pm 1, \ldots, \pm 1, 1, 1, \ldots, 1\rangle\} \quad \text{and}$$
$$S_N^- = \{\langle \pm 1, \ldots, \pm 1, -1, 1, \ldots, 1\rangle\},$$

respectively, each with 2^N elements. There is a natural 1-to-1 mapping

$$s = \langle s_1, \ldots, s_N, 1, \ldots, 1\rangle \mapsto \langle s_1, \ldots, s_N, -1, \ldots, 1\rangle = s^-$$

between S_N and S_N^-.

We have that

$$\begin{aligned} \mu(H_0 \mid \theta) &= F(x_1 + \delta/2, x_2 + \delta/2, \ldots, x_d + \delta/2) \\ &= \sum_{s \in S_0} g(s) \cdot F(C(x, \delta, s) \mid \theta) \\ \mu(H_0^- \mid \theta) &= F(x_1 - \delta/2, x_2 + \delta/2, \ldots, x_d + \delta/2) \\ &= -\sum_{s \in S_0^-} g(s) \cdot F(C(x, \delta, s) \mid \theta) \\ \mu(H_1 \mid \theta) &= \mu(H_0 \backslash H_0^- \mid \theta) = \mu(H_0 \mid \theta) - \mu(H_0^- \mid \theta) \\ &= \sum_{s \in S_0} g(s) \cdot F(C(x, \delta, s) \mid \theta) \\ &\quad + \sum_{s \in S_0^-} g(s) \cdot F(C(x, \delta, s) \mid \theta) \\ &= \sum_{s \in S_1} g(s) \cdot F(C(x, \delta, s) \mid \theta) \end{aligned}$$

Assume that

$$\mu(H_N \mid \theta) = \sum_{s \in S_N} g(s) \cdot F(C(x, \delta, s) \mid \theta).$$

The mapping $s \mapsto s^-$ induces a correspondence between vertices of H_N and H_N^-, and the fact that

$$\mu(H_N^- \mid \theta) = -\sum_{s \in S_N^-} g(s) \cdot F(C(x, \delta, s) \mid \theta)$$

follows by noting the change in sign between each $g(s)$ and corresponding $g(s^-)$. Then

$$\begin{aligned} \mu(H_{N+1} \mid \theta) &= \mu(H_N \backslash H_N^- \mid \theta) = \mu(H_N \mid \theta) - \mu(H_N^- \mid \theta) \\ &= \sum_{s \in S_N} g(s) \cdot F(C(x, \delta, s) \mid \theta) \\ &\quad + \sum_{s \in S_N^-} g(s) \cdot F(C(x, \delta, s) \mid \theta) \\ &= \sum_{s \in S_{N+1}} g(s) \cdot F(C(x, \delta, s) \mid \theta) \end{aligned}$$

By mathematical induction, it must be the case that

$$\mu(H_d \mid \theta) = \sum_{s \in S_d} g(s) \cdot F(C(x, \delta, s) \mid \theta).$$

But $S_d = S$ and the theorem is proved.

Approximating MLE

Since F is smooth, it has a probability density function defined by

$$f(x \mid \theta) \triangleq \lim_{\delta \to 0} \frac{\mu(H(x, \delta) \mid \theta)}{\delta^d}.$$

In general, this is intractable; i.e., a closed-form expression is not convenient. However, since μ can be evaluated, a numerical approximation $f(x \mid \theta, \delta)$ for small δ is readily available by applying the theorem. Therefore, the log-likelihood of a set of data $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ can be approximated by

$$\mathbf{L}(\theta) \approx \sum_i \ln(f(x^{(i)} \mid \theta, \delta))$$

for suitably chosen δ.

This allows the application of standard Maximum Likelihood Estimation techniques.

Sampling

While ML estimation carries its own asymptotic results for assessing standard errors, it might be useful to conduct simulation experiments to bolster these results. A form of rejection sampling is outlined here. Say the goal is to sample n points from the distribution defined by $F(\cdot \mid \theta)$.

Step 1 Choose $N \gg n$ points $x^{(i)}$ uniformly from Ω.

Step 2 For each i, calculate $\phi_i = f(x^{(i)} \mid \theta, \delta)$. Let $m = \max\{\phi_i\}$ and $M = (\Sigma\phi_i)/m$. If $M < n$ then go back to step 1 and choose a larger N.

Step 3 Sample n points without replacement from the discrete distribution $\{\langle x^{(i)}, p_i\rangle\}$ where $p_i = n * \phi_i/(\Sigma\phi_i)$. If you make sure the $x^{(i)}$ are in random order, e.g., the order in which they were first sampled, you can use the following systematic sampling procedure:

Sample := {}. Sp := 0. i := 0.

Repeat

$i := i + 1$.

If the interval (Sp, Sp+ pi] contains an integer, then Sample := Sample $\cup \{x^{(i)}\}$.

$Sp := Sp + p_i$.

Until $Sp = n$.

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