

Reserving

A Response to Dahms's Discussion Published in Variance Vol. 14, Issue 1, 2021

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In this letter we respond to the comments provided in Dahms (2021) presenting alternative views on the model proposed in Siegenthaler (2017) for describing the evolution of ultimate estimates. More specifically we question the simple example provided by Dahms which may not lead to any contradiction with respect to the Siegenthaler formulas for quantifying the prediction uncertainties. Moreover, we show that it is not appropriate to restrict themselves to the assumption $g_j = 1$ in particular when considering ultimate estimates which might be affected by (underlying) model risk.

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1. ANALYZING THE SIMPLE EXAMPLE

For simplicity we consider the special case $I = 3$ and $J = 2$. Using Dahms's notation (see Dahms [2021]) the ultimate estimates $(\hat{U}_{i,j})$, $i \in \{0, \dots, I\}$, $j \in \{0, \dots, J\}$, are denoted as in Figure 1, where those known at time 3 are marked in bold.

Dahms's simple example proposes to consider the ultimate estimates displayed in Figure 2, where $\{U_0, U_1, U_2, U_3\}$ are independent with

$$E[U_i] = g, \quad i \in \{0, 1, 2, 3\}, \quad (1.1)$$

and

$$\text{Var}[U_i] = \sigma^2, \quad i \in \{0, 1, 2, 3\}, \quad (1.2)$$

for some unknown parameters g and σ^2 .

In fact, the ultimate estimates from the simple example can be considered as a realization according to Siegenthaler's model assumptions (see Siegenthaler [2017]) with the following parameters

$$g_0 = 1, \quad \sigma_0^2 = 0, \quad (1.3)$$

Figure 1. Ultimate estimates: Dahms's notation

$$\hat{U}_{i,j} = \hat{U}_i^{i+j}$$

i/j	0	1	$J = 2$
0	$\hat{U}_0^0$	$\hat{U}_0^1$	$\hat{U}_0^2$
1	$\hat{U}_1^1$	$\hat{U}_1^2$	$\hat{U}_1^3$
2	$\hat{U}_2^2$	$\hat{U}_2^3$	$\hat{U}_2^4$
$I = 3$	$\hat{U}_3^3$	$\hat{U}_3^4$	$\hat{U}_3^5$

Figure 2. Ultimate estimates: The simple example

i/j	0	1	$J = 2$
0	1	1	U_0
1	1	1	U_1
2	1	1	U_2
$I = 3$	1	1	U_3

$$g_1 = g, \quad \sigma_1^2 = \sigma^2. \quad (1.4)$$

At time 3, the one-year uncertainty for aggregated accident years is equal to the one-year (and ultimate) uncertainty for accident year 2 and is given by

$$\begin{aligned} & E[(U_2 - 1)^2 | \mathcal{F}^3] \\ &= E[(U_2 - g)^2 + (g - 1)^2 + 2(g - 1)(U_2 - g) | \mathcal{F}^3] \\ &= E[(U_2 - g)^2 | \mathcal{F}^3] + (g - 1)^2 \\ &= \text{Var}(U_2) + (g_1 - 1)^2 \\ &= \sigma_1^2 + (g_1 - 1)^2. \end{aligned} \quad (1.5)$$

Within the Siegenthaler framework, the quantity $E[(U_2 - 1)^2 | \mathcal{F}^3]$ is estimated by

$$\hat{E}[(U_2 - 1)^2 | \mathcal{D}^3] = \hat{\sigma}_1^2 \cdot \hat{U}_2^3 + (\hat{g}_1 - 1)^2 \cdot (\hat{U}_2^3)^2 \quad (1.6)$$

which is fully aligned with the estimator for the one-year prediction uncertainty presented in Siegenthaler (2017) (see Appendix A.1).

Moreover, the parameter estimators are given by

$$\hat{g}_0 = 1, \quad (1.7)$$

$$\hat{g}_1 = \hat{g} = \frac{U_0 + U_1}{2} = \bar{U}, \quad (1.8)$$

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$$\widehat{\sigma}_0^2 = \frac{1}{2} \sum_{i=0}^2 1 \left(\frac{1}{1} - 1 \right)^2 = 0, \quad (1.9)$$

$$\widehat{\sigma}_1^2 = \widehat{\sigma}^2 = \frac{1}{1} \sum_{i=0}^1 1 \left(\frac{U_i}{1} - \bar{U} \right)^2, \quad (1.10)$$

and observe that $\hat{g}$ and $\widehat{\sigma}^2$ are fully consistent with the sample mean and the empirical variance formulas, respectively.

Remark 1. With (1.5) we measure the prediction error at time 3 of predictor $\widehat{U}_2^3 = 1$ for U_2 or, equivalently, the prediction error at time 3 of predictor 0 for $U_2 - \widehat{U}_2^3 = U_2 - 1$. For application in actuarial practice this implies that the one-year risk related to the simple example should be modelled according to a distribution with expected value equal to 0 and variance equal to $\widehat{\sigma}_1^2 + (\hat{g}_1 - 1)^2$.

Remark 2. In Dahms (2021) the author mentions that the ultimate uncertainty for accident year 2 can be estimated by the empirical variance $\widehat{\sigma}^2 = \widehat{\sigma}_1^2$. This proposal can be justified by applying a different measuring approach that considers the “missing compensation” claimed by the author in his article. Indeed we have:

$$\begin{aligned} \widehat{E} \left[\left((U_2 - 1) - \widehat{E}[(U_2 - 1) | \mathcal{D}^3] \right)^2 | \mathcal{D}^3 \right] &= \widehat{E} \left[(U_2 - \hat{g}_1)^2 | \mathcal{D}^3 \right] \\ &= \widehat{\sigma}_1^2 \end{aligned}$$

Furthermore, this approach would imply that the one-year risk related to the simple example should be modelled according to a distribution with expected value equal to $\widehat{E}[(U_2 - 1) | \mathcal{D}^3] = \hat{g}_1 - 1$ and variance equal to $\widehat{\sigma}_1^2$. But note that this handling would not be compliant with the current Solvency standards, which require to model the one-year risk with a distribution centered at 0.

2. ON THE RESTRICTION $g_j = 1$

First recall that the main goal in the development of Siegenthaler's model was to introduce a model that could, **at least approximately**, well describe the evolution of ultimate estimates typically encountered in practice, and then, within that model, derive estimators for the prediction uncertainties. This means that, in case ultimate estimates have been derived according to a stochastic model, we could consider Siegenthaler's model as a surrogate model that approximates the underlying model reasonably closely.

In this respect recall that, despite Solvency requirements prescribing the claim development result to be predicted by zero, for ultimate estimates based on a stochastic model we unfortunately **do not have** (as also highlighted in Dahms [2021])

$$\begin{aligned} E \left[\widehat{U}_{i,j+1} | \mathcal{F}_{i+j} \right] &= \widehat{U}_{i,j}, \\ \text{for all } i &\in \{0, \dots, I\} \text{ and} \\ j &\in \{0, \dots, J-1\}. \end{aligned} \quad (2.1)$$

In other terms, one can show that for an individual accident year i it holds true

$$\begin{aligned} E \left[\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}} | \mathcal{F}_{i+j} \right] &= g_j(\mathcal{F}_{i+j}), \\ \text{for all } j &\in \{0, \dots, J-1\}, \end{aligned} \quad (2.2)$$

for some $\mathcal{F}_{i+j}$ -measurable function $g_j(\mathcal{F}_{i+j}) \neq 1$. Taking the conditional expectation given $\mathcal{D}^{i+j} \subset \mathcal{F}_{i+j}$ on both sides of (2.2) leads to

$$\begin{aligned} E \left[\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}} | \mathcal{D}^{i+j} \right] &= E \left[g_j(\mathcal{F}_{i+j}) | \mathcal{D}^{i+j} \right], \\ \text{for all } j &\in \{0, \dots, J-1\}. \end{aligned} \quad (2.3)$$

Therefore, it is made evident that for an individual accident year i , Siegenthaler's model simply assumes

$$\begin{aligned} E \left[g_j(\mathcal{F}_{i+j}) | \mathcal{D}^{i+j} \right] &= g_j \in \mathbb{R}, \\ \text{for all } j &\in \{0, \dots, J-1\}, \end{aligned} \quad (2.4)$$

whereas for the aggregation over the accident years, it makes use of the modelling idea introduced in Dahms (2012) leading to the assumption

$$\begin{aligned} E \left[\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}} | \mathcal{D}_j^{i+j} \right] &= g_j, \\ \text{for all } i &\in \{0, \dots, I\} \text{ and} \\ j &\in \{0, \dots, J-1\}, \end{aligned} \quad (2.5)$$

which is, for instance, considered acceptable for cumulative payments or reported claim amounts, and basically allows to model the aggregate view without assuming independence between accident years despite implying conditional (given the past information) uncorrelation between data belonging to the same calendar year.

Furthermore, recall that Siegenthaler's model was motivated as a possible solution for quantifying the prediction uncertainties (both one-year and ultimate view) in case the latter could not be quantified in closed form within the stochastic model (if even existing) underlying a specific reserving methodology (see Dal Moro and Lo [2014]). This is particularly the case for reserving methodologies which, from a probabilistic or/and statistical point of view, might not be fully well-defined, such as for instance when considering mixtures of reserving methodologies as often done in actuarial practice. In this respect note that such a mixing approach can be considered admissible since, as mentioned in Dahms (2021), there is some room for interpretation around the mathematical properties that best-estimate ultimates should fulfil and, at each point in time, there might be uncertainty with respect to which reserving methodology is the most appropriate.

Also note that we consider a reserving methodology to be fully well defined (as for instance the most classical chain ladder method) when it is based on a stochastic model for which the model parameter estimators result to be unbiased and the ultimate estimates fulfil the unconditional unbiased property given by

$$\begin{aligned} E \left[\widehat{U}_{i,j} \right] &= E \left[U^i \right], \\ \text{for all } i &\in \{0, \dots, I\} \text{ and} \\ j &\in \{0, \dots, J-1\}. \end{aligned} \quad (2.6)$$

Therefore, given the surrogate connotation of Siegenthaler's model, it is not surprising that there might be inconsistencies when simultaneously assuming Siegenthaler's model and any stochastic model underlying a fully well-defined reserving methodology. Indeed, in the original paper it has already been addressed that Siegenthaler's and

Mack's model (which is the classical model underlying the chain ladder method) cannot simultaneously hold true.

Also, when in a not stringent manner forcing the chain ladder ultimate estimates (or more generally, when considering ultimate estimates for which the unconditional unbiased property holds true) to exactly fulfil Siegenthaler's model assumptions, it has already been demonstrated that it follows $g_j = 1$. Nevertheless, it has been shown that applying the Siegenthaler formulas (for estimating the prediction uncertainties) on ultimate estimates derived from the classical chain ladder methodology we get results which are in line with Mack's (ultimate view) and Merz-Wüthrich's (one-year view) formulas.

With respect to the restriction $g_j = 1$, note that, when considering the fully well-defined chain ladder methodology and when staying stringent within Mack's model assumptions, it holds true (for a single accident year i and according to the original paper notation—see Appendix A.2)

$$E \left[\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} \middle| \mathcal{F}_{i+j} \right] = \left\{ \frac{f_j}{\hat{f}_j^{(i+j)}} \prod_{k=j+1}^{\min(i+j, J-1)} \left[1 + \left(\frac{C_{i+j-k, k}}{\sum_{h=0}^{i+j-k} C_{h, k}} \right) \left(\frac{f_k}{\hat{f}_k^{(i+j)}} - 1 \right) \right] \right\} \hat{U}_{i,j}, \quad (2.7)$$

$=: g_j(\mathcal{F}_{i+j})^{\text{Mack}}$

and remark that since we assume $\hat{f}_k^{(i+j)} = \bar{f}_k$ for $k \geq i+j$ (where $\bar{f}_0, \dots, \bar{f}_{J-1}$ is a set of known a priori expected chain ladder factors) we have $\mathcal{D}^{i+j} = \mathcal{F}_{i+j}$.

Result (2.7) highlights that $E \left[\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} \middle| \mathcal{D}^{i+j} \right] = E \left[g_j(\mathcal{F}_{i+j})^{\text{Mack}} \middle| \mathcal{D}^{i+j} \right] = g_j(\mathcal{F}_{i+j})^{\text{Mack}}$ is generally different from 1.

Moreover, for each fixed development period $j \in \{0, \dots, J-1\}$, $g_j(\mathcal{F}_{i+j})^{\text{Mack}}$ is not constant in i . However, on the long term (i.e., when $i+j$ is sufficiently large) we have

$$\left(\frac{C_{i+j-k, k}}{\sum_{h=0}^{i+j-k} C_{h, k}} \right) \approx 0, \quad k \in \{j+1, \dots, \min(i+j, J-1)\}, \quad (2.8)$$

and, as a consequence, we get the following sufficiently good approximation

$$E \left[\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} \middle| \mathcal{D}^{i+j} \right] = E \left[g_j(\mathcal{F}_{i+j})^{\text{Mack}} \middle| \mathcal{D}^{i+j} \right] \approx \frac{f_j}{\hat{f}_j^{(i+j)}}, \quad \text{for all } j \in \{0, \dots, J-1\}. \quad (2.9)$$

Therefore, from (2.9) we deduce that on the long term the quantities $E \left[\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} \middle| \mathcal{D}^{i+j} \right]$ are, for each fixed development period j , approximately constant in i and therefore compatible with Siegenthaler's model assumptions. Also note that, taking model risk into account, for each fixed develop-

ment period j , the quantity $\frac{f_j}{\hat{f}_j^{(i+j)}}$ does not even necessarily converge to 1 as i increases, since in case the cumulative payments data do not exactly follow Mack's model assumptions, the estimated chain ladder factors $\hat{f}_j^{(i+j)}$ do not necessarily converge to f_j . Thus, when considering Siegenthaler's model as a surrogate model according to the formulation $E \left[\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} \middle| \mathcal{D}^{i+j} \right] = g_j$, we do not need to restrict ourselves to the constraint $g_j = 1$, also in accordance with a possible bias as discussed in Taylor (2003).

More generally, remark that assuming exact unconditional unbiasedness of ultimate estimates in actuarial practice, which might be derived according to different reserving methodologies over time and making use of actuarial judgment, is too strong a requirement. Therefore, when assuming Siegenthaler's model we should again not restrict ourselves to the constraint $g_j = 1$. Indeed, allowing to consider g_j parameters unequal to 1 within Siegenthaler's model is the basic rationale for taking parameter and (underlying) model risk (i.e., the risk of applying the wrong reserving methodology) into consideration.

3. CONCLUSION

We showed that the prediction uncertainties related to the simple example mentioned in Dahms (2021) are in fact fully consistent with the Siegenthaler formulas, and we have highlighted that the model needs not to be restricted to the assumption that $g_j = 1$, in particular when considering ultimate estimates that might be affected by (underlying) model risk. Moreover, the "missing compensation" mentioned by Dahms is fully captured within the parameter and model error terms, and the presented estimators do reflect the uncertainties behind the original ultimate estimates.

Finally, in our opinion Dahms's statement "using chain ladder on estimated ultimates is not a good idea" is questionable in actuarial reserving practice. It is certainly not the perfect design (as it is not for projecting reported claim amounts, because elements of correlation between data belonging to the same calendar year can be detected, too), but the most classical chain ladder reserving methodology (even though it couples accident years) does, on the long term, produce ultimate estimates whose evolution is approximately but fairly well described according to Siegenthaler's model assumptions.

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APPENDICES

APPENDIX A.1

We recall Siegenthaler's formulas extended for also considering the case with only $J \leq I$ development periods:

- One-year view

$$\sum_{i=I-J+1}^I \left[\underbrace{\widehat{\sigma}_{I-i}^2 \cdot \widehat{U}_{i,I-i}}_{\text{"process"}} + \underbrace{(\widehat{g}_{I-i} - 1)^2 \cdot \widehat{U}_{i,I-i}^2}_{\text{"parameter and model"}} \right] + 2 \sum_{I-J+1 \leq i < j \leq I} \underbrace{(\widehat{g}_{I-i} \cdot \widehat{g}_{I-j} - \widehat{g}_{I-i} - \widehat{g}_{I-j} + 1) \cdot \widehat{U}_{i,I-i} \cdot \widehat{U}_{j,I-j}}_{\text{"parameter and model"}}.$$

- Ultimate view

$$\sum_{i=I-J+1}^I \left\{ \underbrace{\left[\sum_{k=I-i}^{J-1} \left(\prod_{j=I-i}^{k-1} \widehat{g}_j \right) \cdot \widehat{\sigma}_k^2 \cdot \left(\prod_{l=k+1}^{J-1} \widehat{g}_l^2 \right) \right] \cdot \widehat{U}_{i,I-i}}_{\text{"process"}} + \underbrace{\left(1 - \prod_{j=I-i}^{J-1} \widehat{g}_j \right)^2 \cdot \widehat{U}_{i,I-i}^2}_{\text{"parameter and model"}} \right\} + 2 \sum_{I-J+1 \leq i < j \leq I} \underbrace{\left(1 - \prod_{k=I-i}^{J-1} \widehat{g}_k \right) \left(1 - \prod_{k=I-j}^{J-1} \widehat{g}_k \right) \cdot \widehat{U}_{i,I-i} \cdot \widehat{U}_{j,I-j}}_{\text{"parameter and model"}}.$$

APPENDIX A.2

Within Mack's model, first note that for fixed $i \in \{0, \dots, I\}$ and $j \in \{0, \dots, J-1\}$ it holds true

$$\begin{aligned} \widehat{U}_{i,j+1} &= C_{i,j+1} \prod_{k=j+1}^{J-1} \widehat{f}_k^{(i+j+1)} \\ &= C_{i,j+1} \prod_{k=j+1}^{\min(i+j, J-1)} \left[\left(1 - \frac{C_{i+j-k,k}}{\sum_{h=0}^{i+j-k} C_{h,k}} \right) \widehat{f}_k^{(i+j)} + \left(\frac{C_{i+j-k,k}}{\sum_{h=0}^{i+j-k} C_{h,k}} \right) \frac{C_{i+j-k,k+1}}{C_{i+j-k,k}} \right] \prod_{k=i+j+1}^{J-1} \widehat{f}_k^{(i+j+1)} \\ &= C_{i,j+1} \prod_{k=j+1}^{\min(i+j, J-1)} \left[\left(1 - \frac{C_{i+j-k,k}}{\sum_{h=0}^{i+j-k} C_{h,k}} \right) \widehat{f}_k^{(i+j)} + \left(\frac{C_{i+j-k,k}}{\sum_{h=0}^{i+j-k} C_{h,k}} \right) \frac{C_{i+j-k,k+1}}{C_{i+j-k,k}} \right] \prod_{k=i+j+1}^{J-1} \widehat{f}_k^{(i+j)}, \end{aligned}$$

where in the last step we made use of the assumption that at time $i+j+1$, the factors $\widehat{f}_k^{(i+j+1)}$ with $k \geq i+j+1$ (for which at time $i+j+1$ there is no observation available) are the same as at time $i+j$.

Therefore, since

$$\left\{ C_{i,j+1}, \left(\frac{C_{i+j-k,k+1}}{C_{i+j-k,k}} \right)_{k=j+1, \dots, \min(i+j, J-1)} \right\} \text{ are independent given } \mathcal{D}^{i+j},$$

it holds true

$$\begin{aligned} &E \left[\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}} \middle| \mathcal{D}^{i+j} \right] \\ &= \frac{E \left[C_{i,j+1} \prod_{k=j+1}^{\min(i+j, J-1)} \left[\left(1 - \frac{C_{i+j-k,k}}{\sum_{h=0}^{i+j-k} C_{h,k}} \right) \widehat{f}_k^{(i+j)} + \left(\frac{C_{i+j-k,k}}{\sum_{h=0}^{i+j-k} C_{h,k}} \right) \frac{C_{i+j-k,k+1}}{C_{i+j-k,k}} \right] \prod_{k=i+j+1}^{J-1} \widehat{f}_k^{(i+j)} \middle| \mathcal{D}^{i+j} \right]}{C_{i,j} \prod_{k=j}^{J-1} \widehat{f}_k^{(i+j)}} \\ &= \frac{f_j \prod_{k=j+1}^{\min(i+j, J-1)} \left[\widehat{f}_k^{(i+j)} + \left(\frac{C_{i+j-k,k}}{\sum_{h=0}^{i+j-k} C_{h,k}} \right) (f_k - \widehat{f}_k^{(i+j)}) \right] \prod_{k=i+j+1}^{J-1} \widehat{f}_k^{(i+j)}}{\prod_{k=j}^{J-1} \widehat{f}_k^{(i+j)}} \\ &= \frac{f_j}{\widehat{f}_j^{(i+j)}} \prod_{k=j+1}^{\min(i+j, J-1)} \left[1 + \left(\frac{C_{i+j-k,k}}{\sum_{h=0}^{i+j-k} C_{h,k}} \right) \left(\frac{f_k}{\widehat{f}_k^{(i+j)}} - 1 \right) \right]. \end{aligned}$$